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Extending the applicability of the local and semilocal convergence of Newton's method

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ABSTRACT

We present a local as well a semilocal convergence analysis for Newton's method in a Banach space setting. Using the same Lipschitz constants as in earlier studies, we extend the applicability of Newton's method as follows: local case: a larger radius is given as well as more precise error estimates on the distances involved. Semilocal case: the convergence domain is extended; the error estimates are tighter and the information on the location of the solution is at least as precise as before. Numerical examples further justify the theoretical results.

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y.

Many problems from Applied Sciences including engineering can be solved by means of finding the solutions of equations in a form like (1.1) using Mathematical Modeling [1,2,5,8,12,14–17]. Except in special cases, the solutions of these equations can be found in closed form. This is the main reason why the most commonly used solution methods are usually iterative. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of the iterative method; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. A very important problem in the study of iterative procedures is the convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypotheses. Another important problem is to find more precise error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$.

Newton's method defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.2}$$

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where x_0 is an initial point, is undoubtedly the most popular method for generating a sequence $\{x_n\}$ approximating x^*. There is a plethora on local as well as semilocal convergence results for Newton's method [1–12,17]. The conditions (C) for the semilocal convergence are:

(C_1) F: $D \subset X \to Y$ is Fréchet differentiable and there exist $x_0 \in D$, $\eta \geq 0$ such that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta$$

(C_2) There exists $L > 0$ such that for each $x, y \in D$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|.$$

In view of (C_2):

(C_3) There exists $L_0 > 0$ such that

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_0\|.$$

Clearly, we have that

$$L_0 \leq L \tag{1.3}$$

and $\frac{L}{L_0}$ can be arbitrarily large [4]. It is worth noticing that (C_3) is not an additional to (C_2) hypothesis, since in practice the computation of constant L involves the computation of constant L_0 as a special case.

Let $U(z, \varrho)$, $\bar{U}(z, \varrho)$ stand, respectively for the open and closed ball in X with center $z \in X$ and radius $\varrho > 0$.

The sufficient convergence criteria for Newton's method using the conditions (C), constants L, L_0 and η given in affine invariant form are:

- Kantorovich [12]

$$h_K = 2L\eta \leq 1. \tag{1.4}$$

- Argyros [4]

$$h_1 = (L_0 + L)\eta \leq 1. \tag{1.5}$$

- Argyros [5]

$$h_2 = \frac{1}{4}\left(L + 4L_0 + \sqrt{L^2 + 8L_0L}\right)\eta \leq 1. \tag{1.6}$$

- Argyros [7,8]

$$h_3 = \frac{1}{4}\left(4L_0 + \sqrt{L_0L + 8L_0^2} + \sqrt{L_0L}\right)\eta \leq 1. \tag{1.7}$$

If $L_0 = L,$ then (1.5)–(1.7) coincide with (1.4). If $L_0 < L$, then

$$h_K \leq 1 \Rightarrow h_1 \leq 1 \Rightarrow h_2 \leq 1 \Rightarrow h_3 \leq 1,$$

but not vice versa. We also have that

$$\begin{aligned} &\frac{h_1}{h_K} \to \frac{1}{2}, \quad \frac{h_2}{h_K} \to \frac{1}{4}, \quad \frac{h_2}{h_1} \to \frac{1}{2} \\ &\frac{h_3}{h_K} \to 0, \quad \frac{h_3}{h_1} \to 0, \quad \frac{h_3}{h_2} \to 0 \\ &\text{as } \frac{L_0}{L} \to 0. \end{aligned} \tag{1.8}$$

Condition (1.8) shows by how many times (at most) the better condition improves the less better condition. Therefore the condition to improve is (1.7). This is done as follows: replace condition (C_2) by

$(C_2)'$ There exists $L_1 > 0$ such that

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L_1\|x - y\|.$$

for each $x, y \in D_0 := U(x_1, \frac{1}{L_0} - \|F'(x_0)^{-1}F(x_0)\|) \cap D,$ where $x_1 = x_0 - F'(x_0)^{-1}F(x_0).$

Denote the conditions (C_1), (C_3) and $(C_2)'$ by $(C)'$. Clearly, we have that

$$L_1 \leq L \tag{1.9}$$

holds in general.