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# New efficient methods for solving nonlinear systems of equations with arbitrary even order[☆]

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ABSTRACT

In 2011, Khattri and Abbasbandy developed an optimal two-step Jarratt-like method for approximating simple roots of a nonlinear equation. We develop their method for solving nonlinear systems of equations. The main feature of the extended methods is that it uses only one LU factorization which preserves and reduces computational complexities. Following this aim, the suggested method is generalized in such a way that we increase the order of convergence but we do not need new LU factorization. Convergence and complexity analysis are provided rigorously. Using some small and large systems, applicability along with some comparisons are illustrated.



## 1. Introduction

Without doubt, solving nonlinear equations and more precisely nonlinear systems of equations has involved great attention of many researcher in science and engineering during the history of mathematics. Although iterative methods, point to point or multipoint ones, for solving single-variable nonlinear equations have been developed and studied very well recently, however, many of these methods cannot be easily extended for solving of nonlinear systems of equations. Due to this fact, there are fewer iterative efficient methods for approximating numerical solution(s) of these systems. This matter has been discussed thoroughly in [1–3] and, more recently, in [4–6] (see also the references therein).

Let the function $\boldsymbol{F}:\boldsymbol{D}\subset\mathbb{R}^n\to\mathbb{R}^n$ has at least, second-order Fréchet derivatives with continuity on an open set $\boldsymbol{D}$. Suppose that the equation $\boldsymbol{F}(\boldsymbol{x})=0$ has a solution $\boldsymbol{x}^*\in\boldsymbol{D}$, that is $\boldsymbol{F}(\boldsymbol{x}^*)=0$, where $\boldsymbol{F}(\boldsymbol{x})=(f_1(\boldsymbol{x}),f_2(\boldsymbol{x}),\dots,f_n(\boldsymbol{x}))^T$, $\boldsymbol{x}=(x_1,x_2,\dots,x_n)^T$, and the coordinate functions $f_i(\boldsymbol{x}), i=1,2,\dots,n$, are real-valued.

It is widely known that Newton's method in several variables [7] could be written as

$$\boldsymbol{x}^{(k+1)}=\boldsymbol{x}^{(k)}-\boldsymbol{F}'(\boldsymbol{x}^{(k)})^{-1}\boldsymbol{F}(\boldsymbol{x}^{(k)}),\quad k=0,1,2,\dots, \tag{1}$$

where $\boldsymbol{x}^{(0)}$ is the initial estimate and $\boldsymbol{F}'(\boldsymbol{x}^{(k)})$ is the Jacobian matrix of the function $\boldsymbol{F}$ evaluated in the $k$th iteration. This method has order of convergence two under certain conditions.

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Another famous scheme for solving nonlinear systems of equations is Jarratt's fourth-order method [2] which is the generalization of the scheme in the scalar case given in [8] as follows

$$\begin{cases} \boldsymbol{y}^{(k)} = \boldsymbol{x}^{(k)} - \frac{2}{3}\boldsymbol{F}'(\boldsymbol{x}^{(k)})^{-1}\boldsymbol{F}(\boldsymbol{x}^{(k)}), \\ \boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \frac{1}{2}\Big[\big(3\boldsymbol{F}'(\boldsymbol{y}^{(k)}) - \boldsymbol{F}'(\boldsymbol{x}^{(k)})\big)^{-1}\big(3\boldsymbol{F}'(\boldsymbol{y}^{(k)}) + \boldsymbol{F}'(\boldsymbol{x}^{(k)})\big)\Big]\boldsymbol{F}'(\boldsymbol{x}^{(k)})^{-1}\boldsymbol{F}(\boldsymbol{x}^{(k)}). \end{cases} \tag{2}$$

Sharma et al. [9] constructed, by composing two weighted Newton step, the following fourth-order method

$$\begin{cases} \boldsymbol{y}^{(k)} = \boldsymbol{x}^{(k)} - \frac{2}{3}\boldsymbol{F}'(\boldsymbol{x}^{(k)})^{-1}\boldsymbol{F}(\boldsymbol{x}^{(k)}), \\ \boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \frac{1}{2}\left[-I + \frac{9}{4}\boldsymbol{F}'(\boldsymbol{y}^{(k)})^{-1}\boldsymbol{F}'(\boldsymbol{x}^{(k)}) + \frac{3}{4}\boldsymbol{F}'(\boldsymbol{x}^{(k)})^{-1}\boldsymbol{F}'(\boldsymbol{y}^{(k)})\right]\boldsymbol{F}'(\boldsymbol{x}^{(k)})^{-1}\boldsymbol{F}(\boldsymbol{x}^{(k)}) \end{cases} \tag{3}$$

Another fourth-order Jarratt-type method has been devised by Babajee et al. [10] as following

$$\begin{cases} \boldsymbol{y}^{(k)} = \boldsymbol{x}^{(k)} - \frac{2}{3}\boldsymbol{F}'(\boldsymbol{x}^{(k)})^{-1}\boldsymbol{F}(\boldsymbol{x}^{(k)}), \\ \boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - 2\left[\boldsymbol{I} - \frac{1}{4}\big(\boldsymbol{F}'(x^{(k)})^{-1}\boldsymbol{F}'(y^{(k)}) - \boldsymbol{I}\big) + \frac{3}{4}\big(\boldsymbol{F}'(\boldsymbol{x}^{(k)})^{-1}\boldsymbol{F}'(\boldsymbol{y}^{(k)}) - \boldsymbol{I}\big)^2\right] \\ \quad \big(\boldsymbol{F}'(\boldsymbol{x}^{(k)}) + \boldsymbol{F}'(\boldsymbol{y}^{(k)})\big)^{-1}\boldsymbol{F}(\boldsymbol{x}^{(k)}). \end{cases} \tag{4}$$

Note that although most of the works emphasizes on the numerical aspects of these iterations, there are two general ways for pursuing this aim analytically. One is based on the well-known $n$-dimensional Taylor expansion [2,11,12] and second is based on the matrix approach, which is so-called as Point of Attraction, introduced first in [7]. We here apply the first case for the sake of simplicity.

In this work, based on results in the scalar case in [13], we extend a new variant of Khattri and Abbasbandi's method with sixth-order convergence in the multidimensional case. It uses two vector-function and two Jacobian matrix evaluations per iteration. Furthermore, we drive a general procedure in such a way that it uses this sixth-order as its predictor using only a new one vector-function evaluation in the corrector step. This is the main idea of this paper. In other words, when we add another step, the order of convergence increases by two units without any new LU decomposition and just consumes one new vector function evaluation. For proving the order of convergence of the new methods, we are going to use the notation introduced in [2]. Let $\boldsymbol{F}: \boldsymbol{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently Fréchet differentiable in an open convex set $\boldsymbol{D}$. By using the $q$th derivative of $\boldsymbol{F}$ at $\boldsymbol{u} \in \mathbb{R}^n$, $q \geq 1$, is the $q$-linear function $\boldsymbol{F}^{(q)}(\boldsymbol{u}) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\boldsymbol{F}^{(q)}(\boldsymbol{u})(v_1, \ldots, v_q) \in \mathbb{R}^n$. It is well known that, for $\boldsymbol{x}^* + \boldsymbol{h} \in \mathbb{R}^n$ lying in a neighborhood of a solution $\boldsymbol{x}^*$ of the nonlinear system $\boldsymbol{F}(\boldsymbol{x}) = 0$, Taylor's expansion can be applied and we have

$$\boldsymbol{F}(\boldsymbol{x}^* + \boldsymbol{h}) = \boldsymbol{F}'(\boldsymbol{x}^*)\left[\boldsymbol{h} + \sum_{q=2}^{p-1}\boldsymbol{C}_q\boldsymbol{h}^q\right] + O(\boldsymbol{h}^p), \tag{5}$$

where $\boldsymbol{C}_q = (1/q!)[\boldsymbol{F}'(\boldsymbol{x}^*)]^{-1}\boldsymbol{F}^{(q)}(\boldsymbol{x}^*)$, $q \geq 2$. We observe that $\boldsymbol{C}_q\boldsymbol{h}^q \in \mathbb{R}^n$ since $\boldsymbol{F}^{(q)}(\boldsymbol{x}^*) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[\boldsymbol{F}'(\boldsymbol{x}^*)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$. In addition, we can express $\boldsymbol{F}'$ as

$$\boldsymbol{F}'(\boldsymbol{x}^* + \boldsymbol{h}) = \boldsymbol{F}'(\boldsymbol{x}^*)\left[\boldsymbol{I} + \sum_{q=2}^{p-1}q\boldsymbol{C}_q\boldsymbol{h}^{q-1}\right] + O(\boldsymbol{h}^p), \tag{6}$$

wherein $\boldsymbol{I}$ is the identity matrix, and $q\boldsymbol{C}_q\boldsymbol{h}^{q-1} \in \mathcal{L}(\mathbb{R}^n)$. Note that in what follows, $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}^*$ is the *error* in the $k$th iteration and $\boldsymbol{e}^{(k+1)} = L\boldsymbol{e}^{(k)^p} + O(\boldsymbol{e}^{(k)^{p+1}})$ is the *error equation*, where $L$ is a $p$-linear function, i.e. $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n, \ldots, \mathbb{R}^n)$ and $p$ is the order of convergence. Observe that $\boldsymbol{e}^{(k)^p} = (\boldsymbol{e}^{(k)}, \boldsymbol{e}^{(k)}, \ldots, \boldsymbol{e}^{(k)})$.

The paper is organized as follows: In Section 2, extensions and developments of Khattri and Abbasbandi's method, [13], for solving nonlinear systems of equations are derived. Furthermore, convergence analysis is established. In Section 3, our primary goal is to develop the general extended method. Numerical test problems and comparisons are illustrated in Section 4. The last section includes some conclusions.