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Sharp bounds for ordinary and signless Laplacian spectral radii of uniform hypergraphs

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ABSTRACT

We give sharp upper bounds for the ordinary spectral radius and signless Laplacian spectral radius of a uniform hypergraph in terms of the average 2-degrees or degrees of vertices, respectively, and we also give a lower bound for the ordinary spectral radius. We also compare these bounds with known ones.

1. Introduction

For positive integers k and n with $k \le n$, a tensor $\mathcal{T} = (T_{i_1 \cdots i_k})$ of order k and dimension n refers to a multidimensional array with complex entries $T_{i_1 \cdots i_k}$ for $i_j \in [n] := \{1, \ldots, n\}$ and $j \in [k]$. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2.

Let $\mathcal{M}$ be a tensor of order $s \ge 2$ and dimension n, and $\mathcal{N}$ a tensor of order $k \ge 1$ and dimension n. The product $\mathcal{MN}$ is the tensor of order $(s-1)(k-1)+1$ and dimension n with entries [10]

$$(\mathcal{MN})_{ij_1 \cdots j_{s-1}} = \sum_{i_2, \ldots, i_s \in [n]} M_{ii_2 \cdots i_s} N_{i_2 j_1} \cdots N_{i_s j_{s-1}},$$

with $i \in [n]$ and $j_1, \ldots, j_{s-1} \in [n]^{k-1}$.

For a tensor $\mathcal{T}$ of order $k \ge 2$ and dimension n and a vector $x = (x_1, \ldots, x_n)^\top$, $\mathcal{T}x$ is an n-dimensional vector whose ith entry is

$$(\mathcal{T}x)_i = \sum_{i_2, \ldots, i_k \in [n]} T_{ii_2 \cdots i_k} x_{i_2} \cdots x_{i_k},$$

where $i \in [n]$. Let $x^{[r]} = (x_1^r, \ldots, x_n^r)^\top$. For some complex ρ, if there is a nonzero n-dimensional vector x such that

$$\mathcal{T}x = \rho x^{[k-1]},$$

then ρ is called an eigenvalue of $\mathcal{T}$, and x an eigenvector of $\mathcal{T}$ corresponding to ρ, see [7,8]. Let $\rho(\mathcal{T})$ be the largest modulus of the eigenvalues of $\mathcal{T}$.

Let $\mathcal{G}$ be a hypergraph with vertex set $V(\mathcal{G}) = [n]$ and edge set $E(\mathcal{G})$, see [1]. If every edge of $\mathcal{G}$ has cardinality k, then we say that $\mathcal{G}$ is a k-uniform hypergraph. Throughout this paper, we consider k-uniform hypergraphs on n vertices with $2 \le k \le n$. A uniform hypergraph is a hypergraph that is k-uniform for some k. For $i \in [n]$, E_i denotes the set of edges of $\mathcal{G}$ containing

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i. The degree of a vertex i in $\mathcal{G}$ is defined as $d_i = |E_i|$. If $d_i = d$ for $i \in V(\mathcal{G})$, then $\mathcal{G}$ is called a regular hypergraph (of degree d). For $i, j \in V(\mathcal{G})$, if there is a sequence of edges $e_1, \ldots, e_r$ such that $i \in e_1$, $j \in e_r$ and $e_s \cap e_{s+1} \neq \emptyset$ for all $s \in [r-1]$, then we say that i and j are connected. A hypergraph is connected if every pair of different vertices of $\mathcal{G}$ is connected.

The adjacency tensor of a k-uniform hypergraph $\mathcal{G}$ on n vertices is defined as the tensor $\mathcal{A}(\mathcal{G})$ of order k and dimension n whose $(i_1 \cdots i_k)$-entry is

$$A_{i_1\cdots i_k} = \begin{cases} \dfrac{1}{(k-1)!} & \text{if } \{i_1, \ldots, i_k\} \in E(\mathcal{G}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}(\mathcal{G})$ be the diagonal tensor of order k and dimension n with its diagonal entry $D_{i\ldots i}$ the degree of vertex i for $i \in [n]$. Then $\mathcal{Q}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) + \mathcal{A}(\mathcal{G})$ is the signless Laplacian tensor of $\mathcal{G}$. We call $\rho(\mathcal{A}(\mathcal{G}))$ the (ordinary) spectral radius of $\mathcal{G}$, which is denoted by $\rho(\mathcal{G})$, and $\rho(\mathcal{Q}(\mathcal{G}))$ the signless Laplacian spectral radius of $\mathcal{G}$, which is denoted by $\mu(\mathcal{G})$.

For a nonnegative tensor $\mathcal{T}$ of order $k \geq 2$ and dimension n, the ith row sum of $\mathcal{T}$ is $r_i(\mathcal{T}) = \sum_{i_2,\ldots,i_k \in [n]} T_{ii_2\cdots i_k}$. If $r_i(\mathcal{T}) > 0$, then the ith average 2-row sum of $\mathcal{T}$ is defined as

$$m_i(\mathcal{T}) = \frac{\sum_{i_2,\ldots,i_k \in [n]} T_{ii_2\cdots i_k} r_{i_2}(\mathcal{T}) \cdots r_{i_k}(\mathcal{T})}{r_i^{k-1}(\mathcal{T})}.$$

Let $\mathcal{G}$ be a k-uniform hypergraph on n vertices. Let $\mathcal{A} = \mathcal{A}(\mathcal{G})$. For $i \in V(\mathcal{G})$ with $d_i > 0$,

$$\begin{aligned} m_i(\mathcal{A}) &= \frac{\sum_{i_2,\ldots,i_k \in [n]} A_{ii_2\cdots i_k} r_{i_2}(\mathcal{A}) \cdots r_{i_k}(\mathcal{A})}{r_i^{k-1}(\mathcal{A})} \\ &= \frac{\sum_{\{i,i_2,\ldots,i_k\} \in E_i} d_{i_2} \cdots d_{i_k}}{d_i^{k-1}}, \end{aligned}$$

which is called the average 2-degree of vertex i of $\mathcal{G}$ (average of degrees of vertices in E_i) [12].

For a k-uniform hypergraph $\mathcal{G}$ with maximum degree Δ, we know that $\rho(\mathcal{G}) \leq \Delta$ [2] and $\mu(\mathcal{G}) \leq 2\Delta$ [8] with either equality when $\mathcal{G}$ is connected if and only if $\mathcal{G}$ is regular (see [9]). Recently, upper bounds for $\rho(\mathcal{G})$ and $\mu(\mathcal{G})$ are given in [12] using degree sequence. In this note, we present sharp upper bounds for $\rho(\mathcal{G})$ and $\mu(\mathcal{G})$ using average 2-degrees or degrees, and we also give a lower bound for $\rho(\mathcal{G})$. We compare these bounds with known bounds by examples.

2. Preliminaries

A nonnegative tensor $\mathcal{T}$ of order $k \geq 2$ dimension n is called weakly irreducible if the associated directed graph $D_{\mathcal{T}}$ of $\mathcal{T}$ is strongly connected, where $D_{\mathcal{T}}$ is the directed graph with vertex set $\{1, \ldots, n\}$ and arc set $\{(i, j) : a_{ii_2\cdots i_k} \neq 0 \text{ for some } i_s = j \text{ with } s = 2, \ldots, k\}$ [3,8].

For an n-dimensional real vector x, let $\|x\|_k = (\sum_{i=1}^n |x_i|^k)^{\frac{1}{k}}$, and if $\|x\|_k = 1$, then we say that x is a unit vector. Let $\mathbb{R}_+^n$ be the set of n-dimensional nonnegative vectors.

Lemma 2.1 [3,11]**.** *Let $\mathcal{T}$ be a nonnegative tensor. Then $\rho(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$ and there is a unit nonnegative eigenvector corresponding to $\rho(\mathcal{T})$. If furthermore $\mathcal{T}$ is weakly irreducible, then there is a unique unit positive eigenvector corresponding to $\rho(\mathcal{T})$.*

Lemma 2.2 [8]**.** *Let $\mathcal{G}$ be a k-uniform hypergraph with n vertices. Then $\rho(\mathcal{G}) = \max\{x^\top(\mathcal{A}(\mathcal{G})x) : x \in \mathbb{R}_+^n, \|x\|_k = 1\}$.*

Lemma 2.3 [6,8]**.** *Let $\mathcal{G}$ be a k-uniform hypergraph. Then $\mathcal{A}(\mathcal{G})$ ($\mathcal{Q}(\mathcal{G})$, respectively) is weakly irreducible if and only if $\mathcal{G}$ is connected.*

A hypergraph $\mathcal{H}$ is a subhypergraph of $\mathcal{G}$ if $V(\mathcal{H}) \subseteq V(\mathcal{G})$ and $E(\mathcal{H}) \subseteq E(\mathcal{G})$.

Lemma 2.4 [2,4]**.** *Let $\mathcal{G}$ be a connected k-uniform hypergraph and $\mathcal{H}$ be a subhypergraph of $\mathcal{G}$. Then $\rho(\mathcal{H}) \leq \rho(\mathcal{G})$ with equality if and only if $\mathcal{H} = \mathcal{G}$.*

For two tensors $\mathcal{M}$ and $\mathcal{N}$ of order $k \geq 2$ and dimension n, if there is an $n \times n$ nonsingular diagonal matrix U such that $\mathcal{N} = U^{-(k-1)}\mathcal{M}U$, then we say that $\mathcal{M}$ and $\mathcal{N}$ are diagonal similar.

Lemma 2.5 [10]**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two diagonal similar tensors of order $k \geq 2$ and dimension n. Then $\mathcal{M}$ and $\mathcal{N}$ have the same real eigenvalues.*

Lemma 2.6 [5,11]**.** *Let $\mathcal{T}$ be a nonnegative tensor of order $k \geq 2$ and dimension n. Then*

$$\min_{1 \leq i \leq n} r_i(\mathcal{T}) \leq \rho(\mathcal{T}) \leq \max_{1 \leq i \leq n} r_i(\mathcal{T}).$$

Moreover, if $\mathcal{T}$ is weakly irreducible, then either equality holds if and only if $r_1(\mathcal{T}) = \cdots = r_n(\mathcal{T})$.