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On the spectrum of the normalized Laplacian of iterated triangulations of graphs

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ABSTRACT

The eigenvalues of the normalized Laplacian of a graph provide information on its topological and structural characteristics and also on some relevant dynamical aspects, specifically in relation to random walks. In this paper we determine the spectra of the normalized Laplacian of iterated triangulations of a generic simple connected graph. As an application, we also find closed-forms for their multiplicative degree-Kirchhoff index, Kemeny's constant and number of spanning trees.

1. Introduction

Important structural and dynamical properties of networked systems can be obtained from the eigenvalues and eigenvectors of matrices associated to their graph representations. The spectra of the adjacency, Laplacian and normalized Laplacian matrices of a graph provide information on the diameter, degree distribution, community structure, paths of a given length, local clustering, total number of links, number of spanning trees and many more invariants [6,8,27,28,36]. Also, dynamic aspects of a network, such as its synchronizability and random walk properties, can be obtained from the eigenvalues of the Laplacian and normalized Laplacian matrices from which it is possible to calculate some interesting graph invariants like the Estrada index and the Laplacian energy [2,4,12,17,22].

In the last years, there has been a particular interest in the study of the eigenvalues and eigenvectors of the normalized Laplacian matrix, since many measures for random walks on a network are linked to them. These include the hitting time, mixing time and Kemeny's constant which can be considered a measure of the efficiency of navigation on the network, see [19,21,24,26,41].

This paper is organized as follows. First, in Section 2, we recall an operation called triangulation that can be applied to any simple connected graph. In Section 3, we determine the spectra of the normalized Laplacians of iterated triangulations of any simple connected graph and we discuss their structure. Finally, in Section 4, we use the results found to calculate three significant invariants for an iterated triangulation of a graph: the multiplicative degree-Kirchhoff index, its Kemeny's constant and the number of spanning trees.

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2. Preliminaries

Let $G(V, E)$ be any simple connected graph with vertex set V and edge set E. Let $N = N_0 = |V|$ denote the number of vertices of G and $E_0 = |E|$ its number of edges.

We consider a graph operation known in different contexts as Henneberg 1 move [18], 0-extension [30], vertex addition [11, p. 63] and triangulation [33]. Here we will use the latter name as it is more common in the recent literature.

Definition 2.1. The triangulation of G, denoted by $\tau(G)$, is the graph obtained by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it.

We denote $\tau^0(G) = G$. The n-triangulation of G is obtained through the iteration $\tau^n(G) = \tau(\tau^{n-1}(G))$ and N_n and E_n will be the total number of vertices and edges of $\tau^n(G)$.

From this definition clearly $E_n = 3E_{n-1}$ and $N_n = N_{n-1} + E_{n-1}$ and thus we have:

$$E_n = 3^n E_0, \qquad N_n = N_0 + \frac{3^n - 1}{2} E_0. \tag{1}$$

Fig. 1 shows an example of iterated triangulations where the initial graph is K_3 or the triangle graph. In this particular case, the resulting graph is known as scale-free pseudofractal graph [14] and it exhibits a scale-free and small-world topology. The study of its structural and dynamic properties has produced an abundant literature, since it is a good deterministic model for many real-life networks, see [9,10,29,40,42] and references therein.

The triangulation operator τ turns each edge of a graph into a triangle, i.e. a cycle of length 3. As we will see in the next section, this property has a relevant impact on the structure of the spectrum of the normalized Laplacian matrix of $\tau^n(G)$.

We label the vertices of $\tau^n(G)$ from 1 to N_n. Let d_i be the degree of vertex i of $\tau^n(G)$, then $D_n = \operatorname{diag}(d_1, d_2, \ldots, d_{N_n})$ denotes its diagonal degree matrix and A_n the adjacency matrix, defined as a matrix with the (i, j)-entry equal to 1 if vertices i and j are adjacent and 0 otherwise.

The Laplacian matrix of $\tau^n(G)$ is $L_n = D_n - A_n$. The probability transition matrix for random walks on $\tau^n(G)$ or Markov matrix is $M_n = D_n^{-1} A_n$. M_n can be normalized to obtain a symmetric matrix P_n.

$$P_n = D_n^{-\frac{1}{2}} A_n D_n^{-\frac{1}{2}} = D_n^{\frac{1}{2}} M_n D_n^{-\frac{1}{2}}. \tag{2}$$

The (i, j)th entry of P_n is $P_n(i, j) = \frac{A_n(i,j)}{\sqrt{d_i d_j}}$. Matrices P_n and M_n share the same eigenvalue spectrum.

Definition 2.2. The normalized Laplacian matrix of $\tau^n(G)$ is

$$\mathcal{L}_n = I - D_n^{\frac{1}{2}} M_n D_n^{-\frac{1}{2}} = I - P_n, \tag{3}$$

where I is the identity matrix with the same order as P_n.

We denote the spectrum of $\mathcal{L}_n$ by $\sigma_n = \{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{N_n}^{(n)}\}$ where $0 = \lambda_1^{(n)} < \lambda_2^{(n)} \leqslant \cdots \leqslant \lambda_{N_n-1}^{(n)} \leqslant \lambda_{N_n}^{(n)} \leqslant 2$. The zero eigenvalue is unique, due to the existence of a stationary distribution for random walks. We denote the multiplicity of λ_i as $m_{\mathcal{L}_n}(\lambda_i)$. Eq. (3) shows a one-to-one correspondence between the spectra of $\mathcal{L}_n$ and P_n.

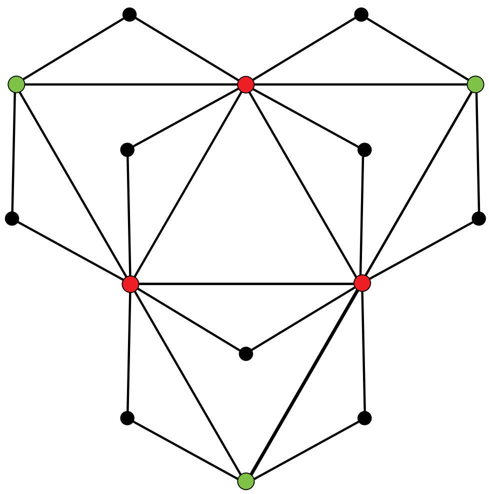

Fig. 1. An example of $\tau^2(G)$. Red vertices denote the initial K_3 vertices while green and black vertices have been introduced to obtain $\tau(K_3)$ and $\tau^2(K_3)$, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).