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# Path integration for real options

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ABSTRACT

Real options were firstly formulated by using traditional financial option models; however, an investor can confront in practice with exotic dynamics. Nowadays, approaches based on simulations have been gaining relevance for solving complex options. This paper proposes the application of the path integral approach (PI) to multivariate real option problems. We discuss the viability of the proposal by a mathematical analysis of the problem and an application to a case study of control chart decision (CCD). The proposal is compared with the traditional approaches for solving real option problems. The results present the proposal as a competitive alternative for the simulation in low dimensional problems.



## 1. Introduction

Real options (RO) can be traced to Myers [1], who first defined investments in real assets as mere options. Hence, the RO approach emerges from the idea of applying financial option appraisal theory to capital investment projects. However, financial options are mainly based on contracts, and conversely, RO are intrinsic features of strategic investments, which must be clearly identified and specified [2]. Several methods were developed to value financial options but their direct applications into the RO setting are conditioned to the particular characteristics of each particular problem. In this sense, the best-known approaches in option valuation, with several possible variants, are: partial differential equations [3], lattice methods [4] and Monte Carlo simulation [5].

The partial differential equation (PDE) approach could be non trivial for RO problems due to its discontinuous formulation. In practice, most RO could have characteristics of American options coupled with several flexibilities and restrictions. The lattice approach is much more simpler than the PDE approach [4]. Its efficiency makes it a proficient appraisal for many RO problems. However, it still has some limitations, such as the difficulty of direct application to any stochastic dynamic. Finally, Monte Carlo-based methods, that could be considered the most versatile approach, always show a stochastic sampling error and a frequent lack of computational efficiency. Besides the above mentioned approaches, a less traditional one is the Path Integral (PI); which have been introduced in Feynman and Hibbs [6]. This approach was born in the context of the quantum mechanics and it has been applied recently to option pricing [7–9]. However, those applications of the PI are restricted to specific dynamics and usually pursue *ad hoc* formulas.

Within this paper, a numerical procedure for RO valuation based on the PI approach is analyzed. A general formulation is applied with the aim to include a wide range of dynamics and possible RO. An additional goal in the application of this approach consists in proposing an approach in order to seize an intermediate trade-off between the computational efficiency of

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the lattice methods and the versatility of the Monte Carlo methods. In this context, this alternative is similar to lattice methods but with a significant difference: the discretization of the stochastic variables is avoided by using an interpolating or regressing procedure in order to ensure continuous stochastic variables. Thus, time is discretized by using numerical schemes applied to the corresponding equations, which defines the stochastic dynamics of the problem. Therefore, a typical dynamic programming is applied but considering infinite points within the stochastic variable interval rather than a finite number of states. The feasibility of this approach is not limited by the number of variables or a specific stochastic model, but by the requirement to ensure the existence of a system of stochastic differential equations with a unique solution that characterizes the RO problem and a numerical discretization which transforms the system in a stochastic recursive equation such that each path has a continuous density function almost everywhere.

This paper formulates an application of the PI approach and its implementation in a problem which considers stochastic variables for both European and American RO. The results and efficiency of the approach are presented by a case study about a quality control chart problem [10]. We compared the numerical results of the proposal to the Black–Scholes formula, binomial method, and Monte Carlo simulation results for one variable of European options. In addition, the results of the binomial and least square Monte Carlo (LSM) [11] methods are applied for one variable of American options. Finally, the pentanomial method [12] and standard Monte Carlo simulation results are applied for two variables of European options. Similarly, pentanomial and least square Monte Carlo (LSM) methods for two variable American options are applied.

Within the proposed approach, the Bellman equation formulation is applied under a risk-neutral measure [13]. Thus, the PI can be employed for the numerical calculation of the conditional expectations that arise in the Bellman equations [14]. These calculations present the form of integrals involving density functions. In Montagna et al. [8], the PI is used to the Geometric-Brownian dynamic [15] in the following way for the conditional expectations

$$\mathbb{E}\left[v\left(s(t^{k+1}),t^k\right)/s(t^k)\right]=\int h_{s(t^k)}v\left(e^{z_i},t^k\right)dz_i \tag{1}$$

where $z(t^k)=\ln(s(t^k))$ and $h_{s(t^k)}$ denote the conditional density function of $s(t^{k+1})$ given $s(t^k)$. In this paper, we approximate the original stochastic process by a numerical scheme estimating the conditional expectations. This procedure is analyzed in detail in Section 3.

This paper is organized as follows: in Section 2, we introduce the path integration approach algorithm for RO. In Section 3, theorems that generalize the estimation of the conditional expectation and analyze a partial error are introduced. Section 4 discusses the model chosen as a case study. Section 5 shows two implementations. Finally, Section 6 discusses and compares the performance of the proposal with traditional algorithms. The comparisons show that the PI approach is a competitive alternative method and it shows good performance when compared to other methods.

## 2. The path integration approach

Let us consider a system of stochastic differential equations with a unique strong solution as follows [22]:

$$ds(t)=f\left(s\left(t\right),t\right)dt+D\left(s\left(t\right),t\right)dw\left(t\right). \tag{2}$$

where $s, ds, f\in\mathbb{R}^n$, $dw\in\mathbb{R}^m$ corresponds to the differential of a Wiener process $w\in\mathbb{R}^m$ and $D\in\mathbb{R}^{n\times m}$. At a specific time $t_i$, the solution of (2) is denoted by $s(t_i)$.

For solving equation (2) numerically, we consider a uniform grid time discretization on $[t_0,t_f]$ with $\hat{k}$ time intervals such that $\tau=(t_f-t_0)/\hat{k}$ and $t^k=k\tau$. Hence, employing the Euler–Maruyana discretization scheme, it is obtained [19]:

$$s^{k+1}=s^k+\widehat{f}(s^k,k)\tau+\widehat{D}(s^k,k)\triangle w^k, \tag{3}$$

where $k$ is the number of steps, $s^k\in\mathbb{R}^n$ the discretizated vector of stochastic processes at step $k$ which approximates $s(t^k)$, $\Delta w^k$ the differentiation of the Wiener process $w$ at step $k$ defined as $\Delta w^k:=w^{k+1}-w^k$ with $w^k:=w(k\tau)$, $\widehat{f}\in\mathbb{R}^n$ is a column vector of $n$ functions $\hat{f}_i:\mathbb{R}^n\times\mathbb{N}\to\mathbb{R}$. $\widehat{D}$ is a matrix $n\times m$, where each entry is a function $\widehat{D}_{ij}:\mathbb{R}^n\times\mathbb{N}\to\mathbb{R}$.

Considering an option value $v$ depending on the underlying asset $s(t^k)$ and the vector $y$ containing particular constant parameters from the problem (the free interest rate or the strike), the corresponding Bellman equation takes the following form

$$v\left(s(t^k),t^k\right)=max_{i\in I(s(t^k),t^k)}(H_i(s(t^k),y,\mathbb{E}[v(s(t^{k+1}),t^{k+1})/s(t^k)])), \tag{4}$$

where $H_i:\mathbb{R}^{n+\hat{y}+1}\to\mathbb{R}$, $\hat{y}=\dim(y)$, $I(s(t^k))$ represents the set of all possible decisions on the state $s(t^k)$ and $\mathbb{E}\left[v\left(s(t^{k+1}),t^{k+1}\right)/s(t^k)\right]$ is the conditional expectation of $v(s(t^{k+1}))$ under $s(t^k)$. For a fixed $k$, the conditional expectation is approximated by

$$\begin{aligned}\mathbb{E}[v(s(t^{k+1}),t^{k+1})/s(t^k)]&\approx\mathbb{E}[v(s^{k+1},t^{k+1})/s^k]\\&=\int_{\mathbb{R}^n}h_{s^k}(\zeta)v(\zeta,t^{k+1})d\zeta,\\&\approx\int_{U_{s^k}}h_{s^k}(\zeta)v(\zeta,t^{k+1})d\zeta,\end{aligned} \tag{5}$$