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Non-oscillatory solutions for a nonlinear neutral delay differential equation

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ABSTRACT

We study the existence of non-oscillatory bounded solutions for a second-order nonlinear neutral delay differential equation. Some sufficient conditions for the existence of non-oscillatory bounded solutions with positive lower bound or negative upper bound are established, and the non-oscillatory solutions with decay as $t \to \infty$ are discussed.

1. Introduction

This paper is concerned with non-oscillatory bounded solutions of the second-order nonlinear neutral delay differential equation of the form

$$\left(r(t)|z'(t)|^{m-1}z'(t)\right)' + \sum_{i=1}^{n} Q_i(t)f_i(x(t-\sigma_i)) = 0, \quad t \geqslant t_0, \tag{1.1}$$

where $z(t) = x(t) - P(t)x(t-\tau), m \geq 1, \tau > 0, \sigma_i \geq 0, n \geq 1,\ P \in C([t_0,\infty),\mathbb{R}), r, Q_i \in C([t_0,\infty),(0,\infty)),\ f_i$ is a continuous and nondecreasing function with $xf_i(x) > 0, x \neq 0, i = 1,2,\ldots,n.$

During the last decades, the oscillation criteria and the existence results of non-oscillatory solutions for various linear and nonlinear differential equations have attracted much attention to mathematicians, see for example [10,1,4,3,11,7,6,5,14,15,13,12,9,2,8,16] and references cited therein. However, as far as second-order nonlinear neutral delay differential equations are concerned, the existence of non-oscillatory bounded solutions has not been extensively studied, due mainly to the technical difficulties arising in its analysis.

It is obvious that solutions with positive lower bound or negative upper bound are non-oscillatory. In 2007, Gong et al. [8] established the existence of bounded solutions with positive lower bound for the case with $m = 1$ and P being a constant unequal to ± 1. Meanwhile, Y. Zhou [9] obtained the existence of bounded solutions with positive lower bound for the case where $m = 1, -1 < -c_1 \leqslant P(t) \leqslant c_2 < 1$ and $c_1 + c_2 < 1$. Moreover, it is quite interesting to consider another class of non-oscillatory solutions, namely those which converge to zero as $t \to \infty$, either positive or negative. It is worthy of mentioning the paper in 2009 by Culáková et al. [10], who investigated the existence of the non-oscillatory positive solutions with decay as $t \to \infty$ under the structure condition

$$\int_{t_0}^{\infty} Q(s)\,\mathrm{d}s = \infty \tag{1.2}$$

and certain condition on P for the case $m, n = 1$. Indeed, they have found such solutions with a decay rate depending on function Q.

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In this paper, we shall continue in this direction the study of the existence of non-oscillatory bounded solutions. We first discuss the non-oscillatory solutions with positive lower bound or negative upper bound. Suppose that the continuous function P satisfies one of the following assumptions including those in [8,9].

(i) $-c_1 \leqslant P(t) \leqslant c_2, 0 \leqslant c_1 < 1 - c_2, t \geqslant t_0$;
(ii) $1 < c_1 \leqslant P(t) \leqslant c_2, 1 < c_1 < c_2, t \geqslant t_0$;
(iii) $-c_1 \leqslant P(t) \leqslant -c_2 < -1, 1 < c_1 < c_2^2, t \geqslant t_0$.

Utilizing Krasnoselskii's Fixed Point Theorem to an integral equation with appropriate parameters, we establish the existence of positive solutions with positive lower bound and negative solutions with negative upper bound.

Secondly, we are interested in obtaining the existence of non-oscillatory solutions with decay without the structure conditions (1.2) on Qs. Under certain condition on P, by utilizing Schauder's Fixed Point Theorem, we obtain the existence of non-oscillatory positive solutions and non-oscillatory negative solutions with the decay rate mainly depending on function P, which is different from those in [10].

This paper is organized as follows. In Section 2, we introduce some necessary preliminaries and state our main results. The proofs of the main results will be given in Section 3. Finally, we provide three examples to illustrate our main results in Section 4.

2. Preliminaries and main results

In this section, we state some preliminaries and present our main results in this paper.

Definition 2.1. By a solution of (1.1) we mean a function $x \in C([t_1 - \delta, \infty), \mathbb{R})$ for some $t_1 \geqslant t_0$, such that $z(t)$ and $r(t)|z'(t)|^{m-1}z'(t)$ are continuously differentiable on $[t_1, \infty)$, and such that (1.1) is satisfied for $t \geqslant t_1$, where $\delta = \max\{\tau, \sigma_1, \sigma_2, \ldots, \sigma_n\}$. Such a solution is said to be non-oscillatory if it has no zeros on $[T, \infty)$ for some $T \geqslant t_1$.

To establish the existence of non-oscillatory bounded solutions of Eq. (1.1), we need the following two fixed point lemmas which are the main tools used in this paper.

Lemma 2.1 (*Krasnoselskii's Fixed Point Theorem [3]*). *Let Ω be a bounded, closed, convex and nonempty subset of a Banach space X, let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then there exists an $x \in \Omega$ such that $S_1x + S_2x = x$.*

Lemma 2.2 (*Schauder's Fixed Point Theorem [3]*). *Let Ω be a closed, convex and nonempty subset of a Banach space X, let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is to say, there exists an $x \in \Omega$ such that $Sx = x$.*

In addition, we also need the following lemma.

Lemma 2.3. *If $A, B \geq 0$, and $m \geq 1$, then*

$$A^{1/m} - B^{1/m} \leqslant |A \pm B|^{1/m} \leqslant A^{1/m} + B^{1/m}.$$

Now we present the main results of the paper. Firstly, we consider the non-oscillatory solutions with positive lower bound or negative upper bound of Eq. (1.1).

Theorem 2.1. *Suppose that*

$$\int_{t_0}^{\infty} \left[\int_s^{\infty} \frac{Q_i(\xi)}{r(s)} \,\mathrm{d}\xi \right]^{1/m} \mathrm{d}s < \infty, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

Further, suppose that there exist nonnegative constants c_1 and c_2 such that one of the following assumptions hold:

(i) $-c_1 \leqslant P(t) \leqslant c_2, 0 \leqslant c_1 < 1 - c_2, t \geqslant t_0$;
(ii) $1 < c_1 \leqslant P(t) \leqslant c_2, 1 < c_1 < c_2, t \geqslant t_0$;
(iii) $-c_1 \leqslant P(t) \leqslant -c_2 < -1, 1 < c_1 < c_2^2, t \geqslant t_0$.

Then Eq. (1.1) admits at least one positive solution with positive lower bound and one negative solution with negative upper bound.

Remark 2.1. Let $m = 1, Q_i \in C([t_0, \infty), \mathbb{R})$ and

$$\int_{t_0}^{\infty} \int_s^{\infty} \frac{|Q_i(\xi)|}{r(s)} \,\mathrm{d}\xi \,\mathrm{d}s < \infty, \quad i = 1, 2, \ldots, n.$$

Then Theorem 2.1 also holds.