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Travelling wave profiles in some models with nonlinear diffusion

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ABSTRACT

We study some properties of the monotone solutions of the boundary value problem

$$(P(u'))' - cu' + f(u) = 0,$$
$$u(-\infty) = 0, \quad u(+\infty) = 1,$$

where f is a continuous function, positive in $(0,1)$ and taking the value zero at 0 and 1, and P may be an increasing homeomorphism of $[0,1)$ or $[0,+\infty)$ onto $[0,+\infty)$. This problem arises when we look for travelling waves for the reaction diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[P\left(\frac{\partial u}{\partial x}\right)\right] + f(u)$$

with the parameter c representing the wave speed.

A possible model for the nonlinear diffusion is the relativistic curvature operator $P(v) = \frac{v}{\sqrt{1-v^2}}$.

The same ideas apply when P is given by the one-dimensional p-Laplacian $P(v) = |v|^{p-2}v$. In this case, an advection term is also considered.

We show that, as for the classical Fisher–Kolmogorov–Petrovski–Piskounov equations, there is an interval of admissible speeds $[c^*, +\infty)$ and we give characterisations of the critical speed c^*. We also present some examples of exact solutions.

1. Introduction

As is well known, the semilinear second order ordinary differential equation

$$u'' - cu' + f(u) = 0,$$

appears in the study of travelling waves for reaction–diffusion equations of Fisher–Kolmogorov–Petrovski–Piskounov (briefly, FKPP) type. Here $c > 0$ is the wave speed and f (a function related to a reaction term) may be of "type A", meaning that f is continuous in $[0,1], f(0) = f(1) = 0$ and $f > 0$ on $(0,1)$. The relevant solutions $u(t)$, defined in the real line, are

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increasing, take values in $[0,1]$ and represent travelling wave profiles. According to the basic theory, there is a minimum admissible speed which is called *critical speed*, and the asymptotic behaviour of the corresponding wave as it approaches the equilibrium 0 allows to distinguish it from those whose speed is not critical.

The problem of travelling waves for FKPP equations has a long history and has originated an impressive literature. We confine ourselves to mention the seminal paper of [10], the deep approach of [2] and more recent works as [11,9] where the reduction to first order singular equations has been extensively and successfully used. Motivation and substantial theory may also be found in the classic monograph of [12].

The purpose of this note is to study a similar kind of problem for quasilinear equations. To be specific, we are concerned with the nonlinear second order differential equation

$$(P(u'))' - cu' + f(u) = 0, \tag{1.1}$$

where P is an increasing homeomorphism of $[0,1)$ or $[0,+\infty)$ onto $[0,+\infty)$ and f is of type A. This is the equation satisfied by travelling waves $u(x+ct)$ to the reaction–diffusion problem in one spatial dimension

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[P\left(\frac{\partial u}{\partial x}\right)\right] + f(u). \tag{1.2}$$

Hence, (1.1) may be seen as an equation for travelling waves in models with general nonlinear diffusion. The Fisher–Kolmogorov–Petrovski–Piskounov equation is obtained setting P equal to the identity and therefore letting diffusion be represented by the one-dimensional Laplacian.

When the domain of P is a bounded interval, the model we have in mind is the so called relativistic curvature operator

$$P(v) = \frac{v}{\sqrt{1-v^2}}. \tag{1.3}$$

The solutions of (1.1) that we look for are heteroclinic connections between the equilibria 0 and 1, that is, they must satisfy the boundary condition

$$u(-\infty) = 0, \quad u(+\infty) = 1, \tag{1.4}$$

in the real line. Given the properties of f and the fact that the meaningful solutions for this model must take values in $[0,1]$, solutions of (1.1)–(1.4) are necessarily monotone. Our aim is to study admissible speeds and to establish results that allow to compute the critical speed.

We should remark that differential equations similar to (1.1) have been recently studied by several authors. For instance, Bereanu et al. [3] have studied periodic solutions to equations involving the nonlinear operator $(P(u'))'$, where P is a rather general increasing homeomorphism of some bounded interval onto $\mathbb{R}$. Torres [13] has studied oscillations for equations as (1.1) with $f(x) = a\sin \pi x$ together with a forcing term in the right-hand side. Thus (1.1) may be seen as a generalisation of the model for a "relativistic pendulum" with friction and, in our case, without forcing. In this paper we propose an additional look at Eq. (1.1), namely as a model for wave profiles, related to the boundary conditions (1.4).

We are also interested in the case where diffusion is given by the one-dimensional p-Laplacian. In that case $P(v) = |v|^{p-2}v, p > 1$. The corresponding Eq. (1.1) has been studied in [7]; here we want to consider the effect of an additional advection term. Advection naturally arises in a variety of physical contexts in combination with diffusion. We shall be dealing with the problem

$$(|u'|^{p-2}u')' - (c - ku)u' + f(u) = 0, \tag{1.5}$$

where $p > 1$ and k is a positive constant. This is the travelling wave equation for

$$\frac{\partial u}{\partial t} - ku\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + f(u). \tag{1.6}$$

We shall also obtain for (1.5) an extension of well known results for the classical FKPP equation. Moreover, we illustrate with rather natural examples the computation of the *critical speed*.

In both situations we shall proceed with the help of a simple reduction method that allows to study monotone solutions of (1.1)–(1.4) via a parametric singular first order problem, as in [5,7]. In particular, we show that the method is quite appropriate to yield exact profiles, some of which correspond to extensions of those already found in the literature.

Models involving quasilinear diffusion of the p-Laplacian type have been extensively considered in the literature. For instance, recently, in [8], the authors used a model of this kind to study travelling waves in convection–diffusion. On the other hand, as long as we know, (1.3) does not seem to have been considered in the context of reaction–diffusion. The novelty with this model is that the quasilinear differential operator has a bounded domain. We believe it is of interest at least for its mathematical meaning, in the sense that in spite of its specificity it inherits many properties of the classic FKPP theory.

2. Reduction to a first order problem

We start with a preliminary remark: