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Convergence analysis of general spectral methods

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ABSTRACT

If a spectral numerical method for solving ordinary or partial differential equations is written as a biinfinite linear system $b = Za$ with a map $Z : \ell_2 \to \ell_2$ that has a continuous inverse, this paper shows that one can discretize the biinfinite system in such a way that the resulting finite linear system $\tilde{b} = \tilde{Z}\tilde{a}$ is uniquely solvable and is unconditionally stable, i.e. the stability can be made to depend on Z only, not on the discretization. Convergence rates of finite approximations $\tilde{b}$ of b then carry over to convergence rates of finite approximations $\tilde{a}$ of a. Spectral convergence is a special case. Some examples are added for illustration.

1. Introduction

In previous papers [1,2], a convergence theory for a fairly general class of linear PDE solving techniques was presented, including unsymmetric kernel-based collocation and meshless Petrov–Galerkin methods. Its basic ingredients were as follows:

1. a well-posed and solvable PDE problem,
2. a trial space that approximates the solution well,
3. a test discretization that is fine enough to guarantee a stability inequality, and finally
4. an optimization routine serving as a solver.

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The final step is necessary because the arising linear systems are not necessarily square and not necessarily solvable, though they have a good approximate solution. This discretization theory was extended to nonlinear problems in a recent paper [3], while the extension to spectral methods is the goal of this paper.

To this end, linear PDE problems and the standard versions of spectral methods (Galerkin, Tau, pseudospectral and Petrov–Galerkin) are presented in Sections 2 and 3, with a common framework described in Section 3.6 that allows a general convergence theory in Section 4 that starts from biinfinite linear systems and considers solvability of discrete subsystems along the steps described above. Among other things, it is proven that well-posed biinfinite linear systems have stable and consistent discretizations, if the latter are properly chosen. The theory is applied to several numerical examples in Section 6.

2. Linear PDE problems

We consider a standard setup for time-independent problems as

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ Bu &= g \quad \text{in } \Gamma := \partial\Omega \end{aligned} \tag{1}$$

with a linear differential operator L and a linear boundary operator B. They map between spaces as

$$\begin{aligned} L &: U \to F \\ B &: U \to G \end{aligned} \tag{2}$$

where U and F are Hilbert spaces of functions on $\overline{\Omega}$ and G is a Hilbert trace space.

The problem (1) is assumed to be *well-posed* in the sense that the operators L and B are bounded maps in (2) and there is a constant C such that

$$\|u\|_U \le C(\|Lu\|_F + \|Bu\|_G) \quad \text{for all } u \in U. \tag{3}$$

For elliptic problems, this usually holds in scales of Sobolev spaces depending on regularity assumptions, but we assume no details here.

3. Spectral methods and others

For all variations of spectral and pseudospectral methods [4–8], the starting point is to write solutions u of (1) as a series expansion

$$u = \sum_{j\in\mathbb{N}} \alpha_j u_j \tag{4}$$

in terms of *trial* functions u_j that in special spectral methods is assumed to be a complete orthonormal system in U. Then,

$$\|u\|_U^2 = \sum_{j\in\mathbb{N}} |\alpha_j|^2 < \infty,$$

and an implicit assumption behind all of this is that the $|\alpha_j|$ decay quickly for increasing j.

If such methods were *meshless*, they should express their trial functions "*entirely in terms of values at nodes*" [9].

Also, the orthogonality of the trial functions is not essential at this point. One can think of finite elements as trial functions as well, but then there is no decay of weights. But since at various places we compare expansions, linear independence will be necessary. An extension to frames is open.

Another common feature of spectral methods and others is that they generate conditions on the α_j by *testing* the residuals $Lu - f$ and $Bu - g$ for solution candidates u. This can be carried out in various ways that we describe now.

3.1. Galerkin methods

Here, the boundary conditions should be homogeneous, and the trial functions should automatically satisfy them. Then, one can drop B completely and change the definition of the spaces U and F accordingly to care for boundary conditions.

Galerkin methods assume $U \subset B$ and then they test the residual $Lu - f$ against the u_j themselves, i.e.

$$\begin{aligned} (Lu - f, u_k)_F &= 0, \\ (Lu, u_k)_F &= (f, u_k)_F, \\ \sum_{j\in\mathbb{N}} \alpha_j \underbrace{(Lu_j, u_k)_F}_{=:L_{jk}} &= \underbrace{(f, u_k)_F}_{\phi_k}, \end{aligned}$$

leading to a biinfinite linear system

$$\sum_{j\in\mathbb{N}} \alpha_j L_{jk} = \phi_k, \quad k \in \mathbb{N} \tag{5}$$

that will appear also in other methods to follow below.