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Recursive computation of generalised Zernike polynomials

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ABSTRACT

An algorithmic approach for generating generalised Zernike polynomials by differential operators and connection matrices is proposed. This is done by introducing a new order of generalised Zernike polynomials such that it collects all the polynomials of the same total degree in a column vector. The connection matrices between these column vectors composed by the generalised Zernike polynomials and a family of polynomials generated by a Rodrigues formula are given explicitly. This yields a Rodrigues type formula for the generalised Zernike polynomials themselves with properly defined differential operators. Another consequence of our approach is the fact that the generalised Zernike polynomials obey a rather simple partial differential equation. We recall also how to define Hermite–Zernike polynomials.

1. Introduction

In this paper we establish a recursive method for computing the generalised Zernike polynomials which are known to be orthogonal on the unit disc of $\mathbb{R}^2$ with respect to the weight function

$$\rho(x, y; \lambda) = (1 - x^2 - y^2)^\lambda, \quad \lambda > -1. \tag{1}$$

The use of Zernike polynomials [1] for describing the classical aberrations of an optical system is well known [2–4]. There have been many other applications, such as to describe the statistical strength of aberrations produced by atmospheric turbulence, atmospheric thermal blooming effects, optical testing, opthalmic optics, corneal topography, interferometer measurements, ocular aberrometry, just to mention a few of them.

The main difficulties when dealing with Zernike polynomials come from the different ordering used in different sources in the literature. The most common is the Noll ordering [5], but Wyant and Creath [6] and Malacara [7] suggest different ones. In general, these orderings preserve the radial power increasing order (that is, the n index), but differ in the order of

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the m term (that is, the angular term). Other orderings motivated by the desire to match certain boundary conditions can be found in [8,9].

In this paper we propose an algorithmic approach for generating recursively the real generalised Zernike polynomials defined as a product of angular functions and radial polynomials

$$Z_{m,j}(\varrho,\theta;\lambda)=\begin{cases} R_m^0(\varrho;\lambda), & m=[j/2],\\ R_m^{m-[j/2]}(\varrho;\lambda)\cos\left(\theta\,(m-[j/2])\right), & m-[j/2]>0,\ j+m^2 \text{ odd},\\ R_m^{m-[j/2]}(\varrho;\lambda)\sin\left(\theta\,(m-[j/2])\right), & m-[j/2]>0,\ j+m^2 \text{ even},\end{cases} \tag{2}$$

for $0\le j\le 2m$, where $0\le\varrho<1$, $0\le\theta<2\pi$, $[x]$ denotes the integer part of x, and ordered in accordance with their total degree. The radial part of $Z_{m,j}(\varrho,\theta;\lambda)$ is

$$R_n^m(\varrho;\lambda)=\sum_{s=0}^{n-m}\frac{(-1)^s(\lambda-m+n+1)_n\varrho^{-m+2n-2s}(\lambda+n+1)_{-m+n-s}}{s!(n-s)!\binom{2n-m}{n-m}(-m+n-s)!}, \tag{3}$$

where $0\le m\le n$, $\lambda>-1$ and $(A)_s=A(A+1)(A+s-1)$, $(A)_0=1$, denotes the Pochhammer symbol. For $\lambda=0$ they coincide with Zernike polynomials [1]. Moreover, these generalised Zernike polynomials are the real and imaginary parts of those in complex variables introduced in [10] and applied in quantum optics in [11]. Operational formulas and generating functions for these complex generalised Zernike polynomials have been obtained in [12].

In order to build the desired algorithm, consider first the column vector of all generalised Zernike polynomials of a fixed total degree s ordered as follows. For odd degree $s=2p+1$, the corresponding vectors of $2p+2$ polynomials are

$$\mathbb{Z}_1^\lambda=\begin{pmatrix} Z_{1,0}(\varrho,\theta;\lambda)\\ Z_{1,1}(\varrho,\theta;\lambda)\end{pmatrix},\qquad \mathbb{Z}_3^\lambda=\begin{pmatrix} Z_{2,2}(\varrho,\theta;\lambda)\\ Z_{2,3}(\varrho,\theta;\lambda)\\ Z_{3,0}(\varrho,\theta;\lambda)\\ Z_{3,1}(\varrho,\theta;\lambda)\end{pmatrix},$$

$$\mathbb{Z}_{2p+1}^\lambda=\begin{pmatrix} Z_{p+1,2p}(\varrho,\theta;\lambda)\\ Z_{p+1,2p+1}(\varrho,\theta;\lambda)\\ Z_{p+2,2p-2}(\varrho,\theta;\lambda)\\ Z_{p+2,2p-1}(\varrho,\theta;\lambda)\\ \cdots\\ Z_{2p+1,0}(\varrho,\theta;\lambda)\\ Z_{2p+1,1}(\varrho,\theta;\lambda)\end{pmatrix},\quad p\ge 2,$$

and for even degree $s=2p$, the vectors of $2p+1$ generalised Zernike polynomials are

$$\mathbb{Z}_2^\lambda=\begin{pmatrix} Z_{1,2}(\varrho,\theta,\lambda)\\ Z_{2,1}(\varrho,\theta,\lambda)\\ Z_{2,0}(\varrho,\theta,\lambda)\end{pmatrix},\qquad \mathbb{Z}_{2p}^\lambda=\begin{pmatrix} Z_{p,2p}(\varrho,\theta;\lambda)\\ Z_{p+1,2p-2}(\varrho,\theta;\lambda)\\ Z_{p+1,2p-1}(\varrho,\theta;\lambda)\\ Z_{p+2,2p-4}(\varrho,\theta;\lambda)\\ Z_{p+2,2p-3}(\varrho,\theta;\lambda)\\ \cdots\\ Z_{2p,0}(\varrho,\theta;\lambda)\\ Z_{2p,1}(\varrho,\theta;\lambda)\end{pmatrix},\quad p\ge 2.$$

We denote by $(\mathbb{Z}_s^\lambda)_i$ the ith polynomial in the column vector $\mathbb{Z}_s^\lambda$. Notice that $\mathbb{Z}_s^\lambda$ contains the $s+1$ generalised Zernike polynomials of total degree s. Thus, we have for $1\le k\le s+1$ and odd s,

$$\left(\mathbb{Z}_s^\lambda\right)_k=\begin{cases} Z_{(s+k)/2,s-k}(\varrho,\theta;\lambda), & k=1,3,5,\ldots,s-2,s,\\ Z_{(s+k-1)/2,s-k+2}(\varrho,\theta;\lambda), & k=2,4,6,\ldots,s-1,s+1.\end{cases} \tag{4}$$

Moreover, for even $s\ge 4$

$$\left(\mathbb{Z}_s^\lambda\right)_k=\begin{cases} Z_{s/2,s-k+1}(\varrho,\theta;\lambda), & k=1,\\ Z_{s/2+[k/2],s-k+2}(\varrho,\theta;\lambda), & k=3,5,\ldots,s-1,s+1,\\ Z_{s/2+[k/2],s-k}(\varrho,\theta;\lambda), & k=2,4,6,\ldots,s-2,s.\end{cases} \tag{5}$$

Observe that generalised Zernike polynomials (2) can be written for odd s as

$$\mathbb{Z}_1^\lambda=(\lambda+1)\begin{pmatrix}\mathrm{Re}(P_{1,0}^\lambda(z,z^*))\\ \mathrm{Im}(P_{1,0}^\lambda(z,z^*))\end{pmatrix},\qquad \mathbb{Z}_3^\lambda=\frac{(\lambda+1)_3}{3!}\begin{pmatrix}\mathrm{Im}(P_{2,1}^\lambda(z,z^*))\\ \mathrm{Re}(P_{2,1}^\lambda(z,z^*))\\ \mathrm{Re}(P_{3,0}^\lambda(z,z^*))\\ \mathrm{Im}(P_{3,0}^\lambda(z,z^*))\end{pmatrix}, \tag{6}$$