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Stability of a fourth order bi-parametric family of iterative methods[☆]

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ABSTRACT

In this paper we present a dynamical study of the Ostrowski–Chun family of iterative methods on quadratic polynomials. We will use dynamical tools such as the analysis of fixed and critical points, and the calculation of parameter planes, to find the most stable members of the family. These results have been checked on the unidimensional Bratu's problem.

1. Introduction

The dynamical analysis of a method is becoming a trend in recent publications on iterative methods because it allows us to classify the various iterative formulas, not only from the point of view of its order of convergence, but also analyzing how these formulas behave as a function of the initial estimate that is taken. It also provides valuable information about the stability and reliability of the iterative method. In this sense, Varona in [1] and Amat in [2], among others, described the dynamical behavior of several well-known families of iterative methods. More recently, in [3–7], the authors analyze, under the point of view of complex dynamics, the quantitative behavior of different known iterative methods and families, such as King's or Chebyshev–Halley's. When this kind of study is made, different pathological numerical behaviors appear, such as periodic orbits, attracting fixed points different from the solutions of the problem, etc. A very useful tool to understand the behavior of the different members of a family is the parameter plane, that helps to select the most stable members of the class.

In this paper we study the dynamical behavior of the Ostrowski–Chun (OC) family of methods introduced in [8], a bi-parametric class of iterative predictor–corrector schemes with order of convergence 4 for solving nonlinear equations or

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systems. We will study the dynamics of OC family of methods on a generic quadratic polynomial, analyzing the fixed and critical points of the associated rational operator. With this analysis we will select the members of the family with good numerical properties. Finally we will perform numerical tests to confirm the qualitative results.

Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as the sequence of points:

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}.$$

A z^* is called a fixed point of an operator R if $z_0 = R(z_0)$. The roots of an equation $f(x) = 0$ are fixed points of the operator associated with the iterative method, but fixed points not corresponding to any root of $f(x) = 0$ may appear. These points are called *strange fixed points*. We can classify fixed points according to the absolute value of the derivated operator on them, so, a fixed point z^* can be:

- Attractor, if $|R'(z^*)| < 1$;
- Superattractor, if $|R'(z^*)| = 0$;
- Repulsor, if $|R'(z^*)| > 1$; and
- Parabolic or indifferent, if $|R'(z^*)| = 1$.

The basin of attraction of an attractor z is the set of preimages of any order: $A(z) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z, n \to \infty\}$. The Fatou set of the rational function R, $F(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the Julia set, $J(R)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The presence of attractive strange fixed points can show a dangerous behaviour of the method, as in this case the scheme can converge to a point that is not a solution of the problem to be solved.

The solutions of $R'(z) = 0$ are called critical points of R. A classical result establishes that there is at least one critical point associated with each immediate invariant Fatou component. A detailed reference of dynamical concepts is given by Devaney in [9].

Two known optimal fourth-order methods are Ostrowski' [10] and Chun's [11] schemes, with iterative expressions

$$x_{k+1} = y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)},$$
$$x_{k+1} = y_k - \frac{f(x_k) + 2f(y_k)}{f(x_k)} \frac{f(y_k)}{f'(x_k)},$$

respectively, where y_k is Newton's step.

We can combine these methods to design a new family of iterative methods. In [8] using Newton's method as predictor we constructed a combination of Ostrowski' and Chun's schemes in the form

$$y_k = x_k - \alpha \frac{f(x_k)}{f'(x_k)},$$
$$x_{k+1} = y_k - \left[\frac{f(x_k)}{a_1 f(x_k) + a_2 f(y_k)} + \frac{b_1 f(x_k) + b_2 f(y_k)}{f(x_k)}\right] \frac{f(y_k)}{f'(x_k)}, \tag{1}$$

where α, a_1, a_2, b_1 and $b_2 \in \mathbb{R}$. We found their values so that the order of convergence is at least 4.

Theorem 1 *([8]). Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function at each point of the open interval I such that $\xi \in I$ is a simple root of the nonlinear equation $f(x) = 0$. Then, the sequence $\{x_k\}_{k\geq 0}$ obtained using expression (1) converges to ξ and the local order of convergence is at least four if $\alpha = 1$, $a_2 = a_1^2(b_2 - 2)$, $b_1 = 1 - \frac{1}{a_1}$ and for all a_1 and $b_2 \in \mathbb{R}$, with $a_1 \neq 0$.*

Then, the iterative formula obtained from Ostrowski–Chun's method (OC) is

$$z_{k+1} = y_k - \frac{1}{a_1}\left[\frac{f(z_k)}{f(z_k) + a_1(b_2 - 2)f(y_k)} + \frac{(a_1 - 1)f(z_k) + a_1 b_2 f(y_k)}{f(z_k)}\right] \frac{f(y_k)}{f'(z_k)}, \tag{2}$$

that defines a two-parameter family of optimal methods with order of convergence four. We can define the operator $\bar{K_f}$ corresponding to OC class.

$$\bar{K_f} = y - \frac{1}{a_1}\left[\frac{f(z)}{f(z) + a_1(b_2 - 2)f(y)} + \frac{(a_1 - 1)f(z) + a_1 b_2 f(y)}{f(z)}\right] \frac{f(y)}{f'(z)} \tag{3}$$

being $y = z - \frac{f(z)}{f'(z)}$ where $z \in \hat{\mathbb{C}}$.