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Existence of solution to a local fractional nonlinear differential equation

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ABSTRACT

We prove existence of solution to a local fractional nonlinear differential equation with initial condition. For that we introduce the notion of tube solution.

1. Introduction

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking noninteger order powers of the differentiation and/or integration operators. Even though the term "fractional" is a misnomer, it has been widely accepted for a long time: the term was coined by the famous mathematician Leibniz in 1695 in a letter to L'Hôpital [1]. In the paper *What is a fractional derivative*? [2], Ortigueira and Machado distinguish between local and nonlocal fractional derivatives. Here we are concerned with local operators only. Such local approach to the fractional calculus dates back at least to 1974, to the use of the fractional incremental ratio in [3]. For an overview and recent developments of the local approach to fractional calculus we refer the reader to [4–7] and references therein.

Recently, Khalil et al. introduced a new well-behaved definition of local fractional (noninteger order) derivative, called the conformable fractional derivative [8]. The new calculus is very interesting and is getting an increasing of interest—see [9,10] and references therein. In [11], Abdeljawad proves chain rules, exponential functions, Gronwall's inequality, fractional integration by parts, Taylor power series expansions and Laplace transforms for the conformable fractional calculus. Furthermore, linear differential systems are discussed [11]. In [12], Batarfi et al. obtain the Green function for a conformable fractional linear problem and then introduce the study of nonlinear conformable fractional differential equations. See also [13] where, using the conformable fractional derivative, a second-order conjugate boundary value problem is investigated and utilizing the corresponding positive fractional Green's function and an appropriate fixed point

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theorem, existence of a positive solution is proved. For abstract Cauchy problems of conformable fractional systems see [14]. Here we are concerned with the following problem:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), & t \in [a, b],\ a > 0, \\ x(a) = x_0, \end{cases} \tag{1}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α, $\alpha \in (0, 1)$. For the first time in the literature of conformable fractional calculus, we introduce the notion of tube solution. Such idea of tube solution has been used with success to investigate existence of solutions for ordinary differentiable equations [15,16], delta and nabla differential equations on time scales [17–19], and dynamic inclusions [20]. Roughly speaking, the tube solution method generalizes the method of lower and upper solution [21–24].

The paper is organized as follows. In Section 2, we present the main concepts of the local conformable fractional calculus and we give some useful preliminary results. In Section 3, we prove existence of solution to problem (1) by using the notion of tube solution and Schauder's fixed-point theorem (see Theorem 19). We end with Section 4, where an illustrative example is given.

2. Preliminaries

We consider fractional derivatives in the conformable sense [8].

Definition 1 (*Conformable Fractional Derivative [8]*)**.** Let $\alpha \in (0, 1)$ and $f : [0, \infty) \to \mathbb{R}$. The conformable fractional derivative of f of order α is defined by $T_\alpha(f)(t) := \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$ for all $t > 0$. Often, we write $f^{(\alpha)}$ instead of $T_\alpha(f)$ to denote the conformable fractional derivative of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say that f is α-differentiable. If f is α-differentiable in some $t \in (0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then we define $f^{(\alpha)}(0) := \lim_{t \to 0^+} f^{(\alpha)}(t)$.

Theorem 2 (*[8]*)**.** *Let $\alpha \in (0, 1]$ and assume f, g to be α-differentiable. Then,*

(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$ *for all* $a, b \in \mathbb{R}$*;*
(2) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$*;*
(3) $T_\alpha(f/g) = (gT_\alpha(f) - fT_\alpha(g))/g^2$.

If, in addition, f is differentiable at a point $t > 0$, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Remark 3. From Theorem 2 it follows that if $f \in C^1$, then one has

$$\lim_{\alpha \to 1} T_\alpha(f)(t) = f'(t)$$

and

$$\lim_{\alpha \to 0} T_\alpha(f)(t) = tf'(t). \tag{2}$$

So $T_\alpha(f)$ is "conformable" in the sense it coincides with f' in the case $\alpha \to 1$ and satisfies similar properties to the integer-order calculus. Note that the property $\lim_{\alpha \to 0} T_\alpha(f) \neq f$ is not uncommon in fractional calculus, both for local and nonlocal operators: see, e.g., the local fractional derivative of [25,26], for which property (2) also holds [27]; and the classical nonlocal Marchaud fractional derivative, which is zero when $\alpha \to 0$ [1]. Note, however, that we only have $T_\alpha(f)(t) = t^{1-\alpha}f'(t)$ in case f is differentiable. If one considers a function that is not differentiable at a point t, then the conformable derivative is not $t^{1-\alpha}f'(t)$. For applications we refer the reader to [10].

Example 4. Let $\alpha \in (0, 1]$. Functions $f(t) = t^p$, $p \in \mathbb{R}$, $g(t) \equiv \lambda$, $\lambda \in \mathbb{R}$, $h(t) = e^{ct}$, $c \in \mathbb{R}$, and $\beta(t) = e^{\frac{1}{\alpha}t^\alpha}$, are α-differentiable with conformable fractional derivatives of order α given by

(1) $T_\alpha(f)(t) = pt^{p-\alpha}$;
(2) $T_\alpha(g)(t) = 0$;
(3) $T_\alpha(h)(t) = ct^{1-\alpha}e^{ct}$;
(4) $T_\alpha(\beta)(t) = e^{\frac{1}{\alpha}t^\alpha}$.

Remark 5. Differentiability implies α-differentiability but the contrary is not true: a nondifferentiable function can be α-differentiable. For a discussion of this issue see [8].