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The heat flux identification problem for a nonlinear parabolic equation in 2D

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ABSTRACT

We consider the heat flux identification problem (HFIP) based on the boundary measurements for a nonlinear parabolic equation in 2-dimensional space. The standard linearization algorithm is applied to the nonlinear direct problem. The method of Conjugate Gradient Algorithm, based on the gradient formula for the cost functional, is then proposed for numerical solution of the inverse heat flux problem. Numerical analysis of the algorithm applied to the inverse problem in typical classes of flux functions is presented. Computational results, obtained for random noisy output data, indicate how the iteration number of the Conjugate Gradient Algorithm can be estimated. Numerical results illustrate bounds of applicability of proposed algorithm, as well as its efficiency and accuracy.

1. Introduction

We study the following inverse problems of determining the unknown flux terms $f := (f_1(x,t), f_2(x,t))$, in the following heat conduction problem

$$\begin{cases} u_t = \nabla(k(|\nabla u|^2)\nabla u) + F(x,t), & (x,t) \in \Omega_T \\ -k(|\nabla u|^2)\dfrac{\partial u}{\partial n} = f_1(x,t), & (x,t) \in \Gamma_1^T, \\ -k(|\nabla u|^2)\dfrac{\partial u}{\partial n} = f_2(x,t), & (x,t) \in \Gamma_2^T \\ u(x,t) = 0, & (x,t) \in \Gamma_3^T \cup \Gamma_4^T, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.1}$$

from the supplementary boundary measurements $h := (h_1(x,t), h_2(x,t))$:

$$\begin{aligned} h_1(x,t) &= u(x,t), \quad (x,t) \in \Gamma_1^T; \\ h_2(x,t) &= u(x,t), \quad (x,t) \in \Gamma_2^T; \end{aligned} \tag{1.2}$$

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where $\Gamma_i^T := \Gamma_i \times (0, T]$, $i = 1, 2, 3, 4$, $x := (x_1, x_2)$, $x \in \Omega$, $\Omega_T := \Omega \times (0, T]$ and $\Omega := (0, \ell_{x_1}) \times (0, \ell_{x_2})$ is assumed to be a bounded simply connected domain with a piecewise smooth boundary $\Gamma := \bigcup_i \Gamma_i$, $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, meas $\Gamma_i \neq 0$, $i = 1, 2, 3, 4$ and $0 < T < \infty$ and $\ell_{x_1}, \ell_{x_2} > 0$.

$$\Gamma_1 := \{0\} \times (0, \ell_{x_2}), \qquad \Gamma_2 := (0, \ell_{x_1}) \times \{0\},$$
$$\Gamma_3 := \{\ell_{x_1}\} \times (0, \ell_{x_2}), \qquad \Gamma_4 := (0, \ell_{x_1}) \times \{\ell_{x_2}\}.$$

The initial temperature $u_0(x)$ and the boundary data $h_1(x, t)$ and $h_2(x, t)$ satisfy the consistency conditions $u_0|_{\Gamma_1} = h_1(x, 0)$ and $u_0|_{\Gamma_2} = h_2(x, 0)$, respectively. For a given flux terms $f_1(x, t)$, $f_2(x, t)$ the problem (1.1) is defined to be the *direct problem*. When the flux term $f := (f_1, f_2)$ needs to be defined, the problem of identifying the unknown f using (1.1)–(1.2) is defined as to be *heat flux identification problem* (HFIP).

Note that in practice the input/output data are obtained from physical experiments and may not be smooth functions. Hence methods based on classical solution of the direct problem cannot be applied for large class of inverse problems. The proposed approach uses the energy method [1] for the solvability analysis of the direct problem. Thus, we are interested in weak (generalized) solution of the parabolic problem (1.1) in $\mathring{V}^{1,0}(\Omega_T) := \{v \in V^{1,0}(\Omega_T) : v(x, t)|_{(\Gamma_3^T \cup \Gamma_4^T)} = 0,\ \forall t \in (0, T]\}$. Here $\mathring{V}^{1,0}$ is the Sobolev space of functions with square integrable gradient ∇u with the norm (see, [2]):

$$\|u\|_{V^{1,0}(\Omega_T)} := \max_{t \in [0,T]} \|u\|_{H^0(\Omega)} + \|\nabla u\|_{H^0(\Omega_T)}.$$

This weak solution $u \in \mathring{V}^{1,0}(\Omega_T)$, with $u(x, 0) = u_0(x)$, satisfies the following integral identity:

$$\frac{1}{2}\int_\Omega [u^2(x,t)]dx + \int_{\Omega_t} k(|\nabla u|^2)|\nabla u|^2\, dxd\tau = \int_{\Omega_t} F(x,\tau)u\, dxd\tau + \frac{1}{2}\int_\Omega [u_0^2(x)]dx + \int_0^T \int_{\Gamma_1} f_1(x,t)u(x,t)dxdt + \int_0^T \int_{\Gamma_2} f_2(x,t)u(x,t)dxdt.$$

For the existence of the unique solution $u(x, t) \in V^{1,0}(\Omega_T)$ we require that the functions $k(\xi)$, $F(x, t)$, $u_0(x)$, satisfy the following conditions [1]:

$$\begin{cases} F(x,t) \in L_2(\Omega_T), \quad f_1(x,t) \in L_2(\Gamma_1^T), \quad f_2(x,t) \in L_2(\Gamma_2^T), \\ u_0(x) \in L_2(\Omega), \quad k(\xi) \in L_\infty(0, \xi^*). \end{cases} \tag{1.3}$$

To investigate solvability conditions for the problem (1.1) in $\mathring{V}^{1,0}$ we add following conditions to function $k(\xi)$:

$$\begin{cases} k(\xi) + 2\xi k'(\xi) \geq \gamma_0 > 0, \quad \xi \in [0, \xi^*] \\ k'(\xi) < 0. \end{cases} \tag{1.4}$$

This study presents a systematic analysis of inverse flux problems aims to estimate as accurately as possible the f, under the overspecified data h at the boundary, given by (1.2). The analysis is based on the proposed variational approach which permits to derive explicitly gradient of the cost functional:

$$J(f) = \int_0^T \int_{\Gamma_1} [u(x,t;f) - h_1(x,t)]^2 dxdt + \int_0^T \int_{\Gamma_2} [u(x,t;f) - h_2(x,t)]^2 dxdt \tag{1.5}$$

corresponding to the above defined problem HFIP. The conjugate gradient method (CGM) with the derived explicit formula for the gradient of the cost functional $J(f)$ is then applied for numerical solution of HFIP.

The paper is organized as follows. In Section 2, we defined quasi solution of the inverse flux problem, introducing the admissible unknown fluxes. In Section 3, we linearize the nonlinear problem to linear one and derived the integral relationship between solutions of the introduced adjoint problems and direct problem. Using these identities we prove the Fréchet differentiability of the cost functional and unicity of the solution in Section 4. In Section 5 the numerical results for the CGM applied to the HFIP are presented for various noise free and noisy output data.

2. The quasi-solution approach of the inverse problems

Let us define the set $\mathcal{W} := \mathcal{F}_1 \times \mathcal{F}_2 \subseteq L_2(\Gamma_1^T) \times L_2(\Gamma_2^T)$ of admissible unknown fluxes $f_1 \in \mathcal{F}_1 \subset L_2(\Gamma_1^T)$ and $f_2 \in \mathcal{F}_1 \subset L_2(\Gamma_2^T)$, which satisfy the following conditions:

$$\begin{cases} -\infty < \underline{F1} < f_1(x,t) < \overline{F1} < \infty, \quad a.e.\ \forall (x,t) \in \Gamma_1^T \\ -\infty < \underline{F2} < f_2(x,t) < \overline{F2} < \infty, \quad a.e.\ \forall (x,t) \in \Gamma_2^T. \end{cases} \tag{2.1}$$

Evidently, $\mathcal{W}$ is a closed and convex subset in $L_2(\Gamma_1^T) \times L_2(\Gamma_2^T)$. For a given element $f \in \mathcal{W}$ we denote by $u(x, t; f) \in \mathring{V}^{1,0}(\Omega_T)$, with $u(x, 0; f) = u_0(x)$, the weak solution of the direct problem (1.1). If the function $u(x, t; f)$ satisfies also the additional condition (1.2), then it will be defined as a strict solution of the problem HFIP, accordingly. In this case, one can introduce