Journal of Computational and Applied Mathematics 312 (2017) 231–239

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Stability of perfectly matched layer regions in generalized finite difference method for wave problems

E. Salete [a,*], J.J. Benito [a], F. Ureña [b], L. Gavete [c], M. Ureña [a], A. García [a]

[a] UNED, ETSII, Madrid, Spain
[b] UCLM, IMACI, Ciudad Real, Spain
[c] UPM, ETSIM, Madrid, Spain

ARTICLE INFO

Article history:
Received 23 July 2015
Received in revised form 22 December 2015

Keywords:
Meshless methods
Generalized finite difference method
Waves propagation
Perfectly matched layer
Stability

ABSTRACT

The generalized finite difference method (GFDM) has been proved to be a good procedure to solve the problem of wave propagation. As other numerical methods, when using a finite model to study an unbounded domain, a way of avoiding wave reflection at the boundaries is needed. Perfectly Matched Layers (PML) absorbing boundary conditions are commonly used.

This paper aims to obtain a Perfectly Matched Layer formulation for the GFDM, for which the stability is guaranteed.

1. Introduction

Meshless methods are, step by step, being considered as an alternative of mesh-based numerical methods, to solve certain problems where these conventional methods present difficulties that meshless methods can avoid.

An important path in the evolution of meshless methods has been the development of the Generalized Finite Difference Method (GFDM), also called meshless finite difference method [1].

The application of GFDM to the solution of the problem of seismic wave propagation using PML absorbing boundary was introduced by Ureña, Benito, Salete and Gavete [2]. These authors have also applied this meshless method to the solution of dynamic problems of beams and plates [3], to solve the advection–diffusion equation [4] and to solve parabolic and hyperbolic equations [5].

Berenger [6] first created a PML for electromagnetic problems in a Finite Element (FE) numerical model. Their equations are based on a field splitting which results in mathematical expressions that cannot be easily manipulated.

Chew and Liu [7] first proposed the PML for elastic waves in solids and proved that reflections are null in a regular elastic medium.

PML has been applied to wave propagation in poroelastic media by Zheng and Huang [8]; to elastic problems by Collino and Tsogka [9], Basu and Chopra [10] and with a FDM scheme Moczo et al. [11], Kirsch [12], Skelton et al. [13], Johnson [14], and for wave equation written as a second-order system in displacements by D. Komatitsch and J. Tromp [15] and Benito et al. [16].

* Corresponding author.
E-mail addresses: esalete@ind.uned.es (E. Salete), jbenito@ind.uned.es (J.J. Benito), fuprieto@terra.com (F. Ureña), lu.gavete@upm.es (L. Gavete), miguelurenya@gmail.com (M. Ureña), angelochurri@gmail.com (A. García).

http://dx.doi.org/10.1016/j.cam.2016.05.027
0377-0427/

This paper shows the application of GFDM to solve these kinds of problems. The scheme used and the analyses of stability and dispersion have been clearly referenced. The use of PML absorbing boundary conditions in the model is also explained and the influence of the loss profile is shown. Finally, the efficiency of the method for solving a variation of Lamb's problem on a domain with a simple topographical feature is illustrated. Also, another example including a hole in the domain is presented.

2. Explicit generalized finite differences schemes for the seismic wave propagation problem

The equations of motion for an elastic, homogeneous, isotropic medium in the domain $\Omega \subset \mathbb{R}^2$ are

$$\rho U_{i,tt} = (\lambda + \mu) U_{j,ji} + \mu U_{i,kk} \tag{1}$$

with the boundary conditions that apply in each case, studied hereinafter.

U_i are the components of the displacement, ρ is the density, λ is the Young's modulus and μ is the Shear modulus (Lamé elastic coefficients).

In the equations throughout this article, index notation and the Einstein notation convention have been used (subindices 1 and 2 should be read as x and y directions).

The aim is to obtain explicit linear expressions for the approximation of partial derivatives at the points of the domain. First of all, an irregular grid or cloud of points is generated in the domain Ω.

On defining the central node with a set of nodes surrounding that node, the star then refers to a group of established nodes in relation to a central node. Every node in the domain has an associated star assigned to it.

This scheme uses the central-difference form for the time derivative.

Following [1,17–19,3], the explicit FD formulae for the second spatial derivatives with second and fourth order approximation ($p = 2, 4$) for the spatial derivatives are obtained

$$[U^0_{i,jk}]_{t=n\Delta t} = -m^{0,p}_{jk}\; {}^n u^0_i + \sum_{l=1}^{N} m^{l,p}_{jk}\; {}^n u^l_i + \Theta[(h_i)^p] \tag{2}$$

where capital letters are used for exact values and small letters are used for approximated values. The superscript n denotes time step, the superscripts 0 and l refer to the central node and the rest of nodes of the star respectively, N is the number of nodes in the star (in this work for the second order approximation $N = 8$ and the star nodes are selected by using the distance criteria [20], and the fourth order approximation $N = 30$ and the star nodes are selected by using the distance criteria) and $h^l_i = x^l_i - x^0_i$.

$m^{0,p}_{jk}$ are the coefficients that multiply the approximate values of the functions U_i at the central node (${}^n u^0_i$) for the time $n\Delta t$ in the generalized finite difference explicit expressions for the space derivatives, and $m^{l,p}_{jk}$ are the coefficients that multiply the approximate values of the functions U_i at the other nodes of the star (${}^n u^l_i$). In all these expression the cross-terms are equal.

The replacement in Eq. (1) of the explicit expressions obtained for the spatial derivatives and the central-difference formula for the time derivatives leads to the explicit difference scheme

$$\begin{aligned} {}^{n+1}u^0_i &= 2\,{}^n u^0_i - {}^{n-1}u^0_i + \frac{(\Delta t)^2}{\rho}\left[(\lambda+\mu)\left(-m^{0,p}_{ji}\; {}^n u^0_j + \sum_{l=1}^{N} m^{l,p}_{ji}\; {}^n u^l_j\right)\right. \\ &\quad \left. + \mu\left(-m^{0,p}_{kk}\; {}^n u^0_i + \sum_{l=1}^{N} m^{l,p}_{kk}\; {}^n u^l_i\right)\right] + \Theta[(\Delta t)^2, (h_i)^p] \end{aligned} \tag{3}$$

where Θ represents the error of the scheme.

3. Stability criterion of the scheme

For the stability analysis the idea is to make a harmonic decomposition of the approximated solution at grid points and at a given time level (n)

$${}^n u^0_j = A\xi^n e^{i\boldsymbol{k}^T\boldsymbol{x_0}}; \qquad {}^n u^l_j = A\xi^n e^{i\boldsymbol{k}^T\boldsymbol{x_l}} \quad \forall j = 1, 2;\ l = 1 \cdots N \tag{4}$$

where i is the imaginary unit, j are the directions of the orthogonal coordinate axis, the vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$ ($l = 1, \ldots, N$) represent the position of the central node and the rest of the nodes of the star respectively, N is the number of nodes in the star, and

$$\xi = e^{-iw\Delta t}; \qquad \boldsymbol{k} = \kappa \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix}. \tag{5}$$