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Numerical solution of the Burgers equation with Neumann boundary noise

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ABSTRACT

In this paper we investigate the numerical solution of the one-dimensional Burgers equation with Neumann boundary noise. For the discretization scheme we use the Galerkin approximation in space and the exponential Euler method in time. The impact of the boundary noise on the solution is discussed in several numerical examples. Moreover, we analyze and illustrate some properties of the stochastic term and study the convergence numerically.

1. Introduction

Stochastic partial differential equations (SPDE) arise naturally due to environmental fluctuations subject to random influences. Under some physical circumstances the physical boundary of the problem is affected by noise. Such models may be interpreted by partial differential equations (PDEs) with random Neumann boundary conditions [1–5]. The first paper which studied evolution problems with boundary noise was a paper [6] by Balakrishnan. Later Sowers [4] investigated general reaction diffusion equation with Neumann type boundary noise. Da Prato and Zabczyk [2] discussed the difference between problems with Dirichlet and Neumann boundary noises, as Dirichlet noise does not lead to sufficiently regular solutions. A related work for parabolic problems with boundary noise can be referred to Brzezniak and Peszat [5]. In this paper we focus on the following Burgers equation subject to Neumann boundary conditions on the interval $(0, l)$ with boundary noise in only one of the Neumann conditions:

$$\begin{cases} \partial_t u = \nu \partial_{xx} u - u \partial_x u, \\ \partial_x u(0, t) = \sigma \dot{\beta}(t), \qquad \partial_x u(l, t) = 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{1}$$

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Here the scalar $\sigma > 0$ denotes the noise strength and $\{\dot{\beta}(t)\}_{t\geq 0}$ is white noise in time, given by the generalized derivative of a real valued Brownian motion $\{\beta(t)\}_{t\geq 0}$. Finally, the scalar $\nu > 0$ denotes the viscosity. Without loss of generality after rescaling space and time we can assume from now on $\nu = 1$.

For our numerical results we are interested in solutions given in a weak or mild sense. Before we give a definition of these concepts, let us first introduce some basic notation. Fix a terminal time $T > 0$ and let (Ω, F, P) be a probability space, equipped with the filtration $(\mathbb{F}_t)_{t\geq 0}$ induced by the driving Brownian motion.

As candidates for solutions we only consider $L^2(0, l)$-valued stochastic processes $u : \Omega \times [0, T] \to L^2(0, l)$ with continuous paths and adapted to the filtration $(\mathbb{F}_t)_{t\geq 0}$ that satisfy

$$\sup_{t\in[0,T]} \left(\mathrm{E}\, |u(t)|^2_{L^2(0,l)} \right) < \infty.$$

According to Da Prato & Zabczyk [2], we consider only solutions u given by the splitting

$$u(x, t) = v(x, t) + w(x, t),$$

where first the process $v(x, t)$ satisfies a linear SPDE (in a weak sense defined later)

$$\begin{cases} \partial_t v = \partial_{xx} v, \quad 0 < x < l,\ t > 0, \\ \partial_x v(0, t) = \sigma \dot{\beta}(t), \qquad \partial_x v(l, t) = 0, \\ v(x, 0) = 0, \end{cases} \tag{2}$$

and secondly $w(y, t)$ solves the random PDE

$$\begin{cases} \partial_t w = \partial_{xx} w - (v + w)\partial_x(v + w), \quad 0 < x < l,\ t > 0, \\ \partial_x w(0, t) = 0, \qquad \partial_x w(l, t) = 0, \\ w(x, 0) = u_0(x). \end{cases} \tag{3}$$

We will see later, that the solution of (3) at least locally in time can be obtained by usual deterministic methods based on fixed point theorems. Moreover, it can be well approximated by PDE solvers (see for example [7]). For the linear SPDE we follow [2] and are interested in the solution of the SPDE (2) in the weak sense, given by an Ornstein–Uhlenbeck process called the stochastic convolution. Note that the derivatives of w in general will not exist. Nevertheless, it is well known, that the existence of a sufficiently regular solution to (2) implies the existence of a solution to (3) and thus (1). See for example [2].

This paper is organized as follows. In Section 2 we formulate the problem and obtain a series expansion of the solution of (2) given as stochastic convolution. Then we describe the numerical method based on the mild formulation of (2) and (3). In Section 3 we give numerical examples of the Burgers equation subject to random Neumann boundary conditions and study how the noise which is strictly localized on the boundary extends immediately to the entire domain. In Section 4 we compute the difference between solutions with $\sigma = 0$ and $\sigma \neq 0$ using different metrics. In the last section we analyze some properties of the stochastic term and illustrate these properties numerically. Finally, we briefly illustrate the rate of convergence of our numerical scheme by numerical experiments. We think it should also be possible to prove the rate of convergence, but due to non-Lipschitz nonlinearities and poor regularity properties of the stochastic convolution, this would be quite involved.

2. Problem formulation

Denote for the length $l > 0$ of the underlying domain by $L^2(0, l)$ (or short L^2) the standard space of square-integrable functions with the standard scalar product $(u, v)_{L^2} = \int_0^l u(x)v(x)dx$. We also use L^p-spaces $L^p(0, l)$ with norm $\|f\|_{L^p} = (\int_0^l |f(x)|^p dx)^{1/p}$, and similarly L^∞-spaces of measurable and essentially bounded functions.

Definition 2.1. An $L^2(0, l)$ valued process $(v(t))_{t\in[0,T]}$ with continuous paths and adapted to the filtration $\mathbb{F}$ is a weak solution of (2) if

$$(v(t), \Psi)_{L^2} = (v_0, \Psi)_{L^2} + \int_0^t (v(s), A\Psi)_{L^2}\, ds + \beta(t)\Psi(0)$$

for all $t \in [0, T]$ and all smooth test functions $\Psi \in C^\infty([0, l])$ satisfying Neumann boundary conditions

$$\frac{\partial \Psi}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = l.$$

Here $A = \partial_{xx}$ is the Laplace operator with the domain of definition $D(A)$, which is given by the subset of the standard Sobolev-space $H^2(0, l)$ of twice weakly differentiable functions with square integrable derivatives satisfying Neumann boundary conditions:

$$D(A) = \left\{ \Psi \in H^2(0, l) : \frac{\partial \Psi}{\partial x}(0) = 0,\ \frac{\partial \Psi}{\partial x}(l) = 0 \right\}.$$