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On an optimal finite element scheme for the advection equation

Argus A. Dunca

Department of Mathematics and Informatics, University POLITEHNICA of Bucharest, Splaiul Independentei 313, RO-060042, Bucharest, Romania

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ABSTRACT

In this report it is presented a numerical finite element scheme for the advection equation that attains the optimal L^2 convergence rate $\mathcal{O}(h^{k+1})$ when order k finite elements are used, improving the order $\mathcal{O}(h^{k+0.5})$ of other previous regularization methods. This result is also confirmed by the numerical test performed in the last section. The scheme assumes unstructured grids, periodic boundary conditions, a constant advection field and a bit (two units) stronger regularity on the exact solution than in the classical (suboptimal) finite element theory.

1. Introduction

In general the $L^\infty(L^2)$-error of the standard finite element Galerkin method for first order hyperbolic systems converges in the order $\mathcal{O}(h^k)$ where h is the mesh size and k the order of the finite elements, see Dupont, [1], one unit less than expected. Optimal convergence has been proved only in some particular cases (such as linear elements or cubic splines on uniform grids and for periodic boundary conditions, see Dupont [1], Thomee and B. Wendroff, [2]).

Various regularization methods have been employed to improve the convergence rate on unstructured grids. In the class of filter based regularization methods, such as the one used here we mention Layton and Connors, [3], Ervin and Jenkins, [4], Dunca and Neda [5]. If periodic boundary conditions are assumed and unstructured grids and order k elements are used, to the author's knowledge the best convergence rate available in the literature is $\mathcal{O}(h^{k+0.5})$, see for example the models in Layton and Connors, [3], or Dunca and Neda [5].

This paper considers a numerical scheme to solve the model advection equation

$$u_t + \vec{a} \cdot \nabla u = f, \quad u(0) = u_0 \tag{1}$$

optimally in case $\vec{a}$ is a constant vector and periodic boundary conditions are assumed. The mesh and the finite element spaces X_h are chosen such that a general approximation property, see inequality (4), holds. The exact solution is not necessarily smooth, but it should be a bit more regular (two powers) than in the classical suboptimal theory.

The algorithm presented herein is based on the idea developed in the papers of Dunca, John and Layton, [6,7], which is that, in some cases, the mean finite element error has a higher convergence rate than the finite element error itself. Here the mean $\overline{v}$ of v is computed using the differential filter (on the fixed length scale $\delta = 1$), see Germano, [8], Dunca and John, [6],

E-mail address: argus_dunca@yahoo.com.

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$\overline{v} = \mathcal{S}(v)$, where

$$\overline{v} - \Delta\overline{v} = v. \tag{2}$$

To approximate optimally the solution u of Eq. (1), we first apply the differential operator $I - \Delta$ to Eq. (1) to get

$$(I-\Delta)u_t + \vec{a}\cdot\nabla(I-\Delta)u = (I-\Delta)f, \qquad (I-\Delta)u(0) = (I-\Delta)u_0.$$

Therefore $(I-\Delta)u$ is the solution w of the problem

$$w_t + \vec{a}\cdot\nabla w = (I-\Delta)f, \qquad w(0) = (I-\Delta)u_0 \tag{3}$$

i.e., $w = (I-\Delta)u$, and therefore, using Eq. (2), we obtain $u = \overline{w}$.

In this regard we may view the solution u of problem (1) as being the exact average $\overline{w}$ of the solution w of problem (3). As such, one expects better convergence rate if, instead of solving directly (and suboptimally) with FEM problem (1), one first solves with FEM problem (3) to get w_h (which is a suboptimal approximation of w) and then filters w_h to obtain $\overline{w_h}^h$. In Section 4 we prove that, if u satisfies several regularity assumptions, then $\overline{w_h}^h$ is indeed an optimal approximation of $\overline{w} = u$, i.e. $\|\overline{w_h}^h - u\|_{L^\infty([0,T],L^2(\Omega))}$ is $\mathcal{O}(h^{k+1})$ where k is the order of the finite elements.

2. Mathematical setting

We let Ω be the $2d$ or $3d$ periodic box. The norm $\|\cdot\|$ will denote the usual L^2 norm on Ω and $(\cdot,\cdot)$ will be the corresponding L^2 inner product on Ω. For a given natural number k, H^k will denote the usual Sobolev space of order k on Ω and $\|\cdot\|_k$ and $|\cdot|_k$ are its usual Sobolev norm and seminorm respectively.

$H^k_\#(\Omega)$ will denote the closure of the smooth, periodic functions defined on Ω in the Sobolev $\|\cdot\|_k$ norm. For $k = 1$ we let $X = H^1_\#(\Omega)$ and for $k = 0$ we let $L^2_\#(\Omega) = H^0_\#(\Omega)$.

In the sequel $X_h \subset X$ will denote a conforming finite element space on a quasi-uniform mesh of size h on Ω satisfying the general approximation assumption that there exists a general constant C such that

$$\|v - v_h\| + h\|\nabla v - \nabla v_h\| \le Ch^{l+1}|v|_{l+1} \tag{4}$$

for some interpolant $v_h \in X_h$ of $v \in X \cap H^{l+1}$, where $1 \le l \le k$.

For $u \in L^2_\#(\Omega)$ its mean $\overline{u} \in H^2_\#(\Omega) \subset L^2_\#(\Omega)$ is defined using the differential filter, see Germano, [8], as the unique solution of the PDE

$$-\Delta\overline{u} + \overline{u} = u \tag{5}$$

with periodic boundary conditions. We let $\mathcal{S} : L^2_\#(\Omega) \to L^2_\#(\Omega)$, $\mathcal{S}u = \overline{u}$, denote the differential filtering operator.

We also let the discrete mean $\overline{u}^h \in X_h$ of u to be the classical FEM approximation of $\overline{u}$, defined by

$$(\nabla\overline{u}^h, \nabla v_h) + (\overline{u}^h, v_h) = (u, v_h)$$

for any $v_h \in X_h$. We let $\mathcal{S}_h : L^2_\#(\Omega) \to L^2_\#(\Omega)$, $\mathcal{S}_h u = \overline{u}^h$, denote the discrete differential filtering operator.

Remark 2.1. One can show, see [6,9,10] that the differential filtering operators $\mathcal{S}$, $\mathcal{S}_h$ are selfadjoint and they satisfy the stability inequality

$$\|\overline{v}\| \le \|v\|, \qquad \|\overline{v}^h\| \le \|v\|, \quad \forall v \in L^2_\#(\Omega). \tag{6}$$

The following known result states the classical FEM convergence rate of the elliptic second order PDE (5), obtained using Céa's lemma and the Aubin–Nitsche duality method, see Brenner and Scott, Theorem 5.7.6 on page 144, [11].

Remark 2.2. For $u \in X$ there holds

$$\|\overline{u} - \overline{u}^h\| + h\|\nabla\overline{u} - \nabla\overline{u}^h\| \le Ch^2\|\overline{u}\|_2 \le Ch^2\|u\|. \tag{7}$$

In case $u \in H^{k-1}_\#(\Omega)$ we have that

$$\|\overline{u} - \overline{u}^h\| + h\|\nabla\overline{u} - \nabla\overline{u}^h\| \le Ch^{k+1}\|\overline{u}\|_{k+1} \le Ch^{k+1}\|u\|_{k-1}. \tag{8}$$

Here C is a general constant not depending on u or h.