



# Determination of a term in the right-hand side of parabolic equations

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## ABSTRACT

The inverse problem of determining a term in the right hand side of parabolic equations from integral observations is investigated. The observations can be regarded as generalized interior point observations which are collected in practice. The problem is then reformulated as a least squares problem in coupling with a Tikhonov regularization term. It is proved that the Tikhonov functional is Fréchet differentiable and a formula for the gradient is derived via an adjoint problem. The variational problem is discretized by the finite element method, the convergence of which is proved. The discretized variational problem is numerically solved by the conjugate gradient method. Some numerical examples are presented for showing the efficiency of the method.

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## 1. Introduction

The problem of determining a term in the right hand side of parabolic equations attracted great attention of many researchers during the last 50 years. Despite a lot of results on the existence, uniqueness and stability estimates of a solution to the problem, its ill-posedness and possible nonlinearity make it not easy and require further investigations. For surveys on the subject, we refer the reader to the books [1–5] and the recent paper [6]. To be more detailed, let  $\Omega \in \mathbb{R}^d$  be a bounded domain with the boundary  $\Gamma$ . Denote the cylinder  $Q := \Omega \times (0, T]$ , where  $T > 0$  and the lateral surface area  $S = \Gamma \times (0, T]$ . Let

$$a_{ij}, \quad i, j \in \{1, 2, \dots, d\}, \quad b \in L^\infty(Q), \quad (1.1)$$

$$a_{ij} = a_{ji}, \quad i, j \in \{1, 2, \dots, d\}, \quad (1.2)$$

$$\lambda \|\xi\|_{\mathbb{R}^d}^2 \leq \sum_{i,j=1}^d a_{ij}(x, t) \xi_i \xi_j \leq \Lambda \|\xi\|_{\mathbb{R}^d}^2, \quad \forall \xi \in \mathbb{R}^d, \quad (1.3)$$

$$0 \leq b(x, t) \leq \mu_1, \quad \text{a.e. in } Q, \quad (1.4)$$

$$u_0 \in L^2(\Omega), \quad \varphi, \psi \in L^2(S), \quad (1.5)$$

$$\lambda \text{ and } \Lambda \text{ are positive constants and } \mu_1 \geq 0. \quad (1.6)$$

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Consider the initial value problem

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) + b(x, t)u = F(x, t), \quad (x, t) \in Q, \quad (1.7)$$

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \quad (1.8)$$

with either the Robin boundary condition

$$\frac{\partial u}{\partial \mathcal{N}} + \sigma u|_S = \varphi \quad \text{on } S \quad (1.9)$$

or the Dirichlet boundary condition

$$u|_S = \psi \quad \text{on } S. \quad (1.10)$$

Here,

$$\frac{\partial u}{\partial \mathcal{N}} \Big|_S := \sum_{i,j=1}^d (a_{ij}(x, t) u_{x_j}) \cos(\nu, x_i) \Big|_S,$$

$\nu$  is the outer normal to  $S$  and  $\sigma \in L^\infty(S)$  which is supposed to be nonnegative everywhere in  $S$ .

The direct problem is that of determining  $u$  when the coefficients of Eq. (1.7) and the data  $u_0$ ,  $\varphi$  (or  $\psi$ ) and  $F$  are given [2,7,8]. The inverse problem is that of identifying the right hand side  $F$  when some additional observations of the solution  $u$  are available. Depending on the structure of  $F$  and observations of  $u$  we have different inverse problems:

- Inverse Problem (IP) 1:  $F(x, t) = f(x, t)h(x, t) + g(x, t)$ ,  $h$  and  $g$  are given. Find  $f(x, t)$ , if  $u$  is given in  $Q$  [9,10].
- IP2:  $F(x, t) = f(x)h(x, t) + g(x, t)$ ,  $h$  and  $g$  are given. Find  $f(x)$ , if  $u(x, T)$  is given, [11–15]. Related inverse problems for nonlinear equations have been studied in [16–18].
- IP2a:  $F(x, t) = f(x)h(x, t) + g(x, t)$ ,  $h$  and  $g$  are given. Find  $f(x)$ , if  $\int_\Omega \omega_1(t)u(x, t)dx$  is given. Here,  $\omega_1$  is in  $L^\infty(0, T)$  and nonnegative. Furthermore,  $\int_0^T \omega_1(t)dt > 0$ . Such an observation is called *integral observation* and it is a generalization of the final observation in IP2, when  $\omega_1$  is an approximation to the delta function at  $t = T$ . The problem of this setting has been studied in [19–24,6,25].
- IP3:  $F(x, t) = f(t)h(x, t) + g(x, t)$ ,  $h$  and  $g$  are given. Find  $f(t)$ , if  $u(x_0, t)$  is given. Here,  $x_0$  is a point in  $\Omega$  [26–28].
- IP3a:  $F(x, t) = f(t)h(x, t) + g(x, t)$ ,  $h$  and  $g$  are given. Find  $f(t)$ , if  $\int_\Omega \omega_2(x)u(x, t)dx$  is given. Here,  $\omega_2 \in L^\infty(\Omega)$  with  $\int_\Omega \omega_2(x)dx > 0$ . See, e.g. [29,30,22].
- IP4:  $F(x, t) = f(x)h(x, t) + g(x, t)$ ,  $h$  and  $g$  are given. Find  $f(x)$  if an additional boundary observation of  $u$ , for example, in case of the Dirichlet boundary condition (1.9), we require the Neumann condition be given in a subset of  $S$ , [31–39,2,40–42]. A similar problem for identifying  $f(t)$  with  $F(x, t) = f(t)h(x, t) + g(x, t)$  has been studied in [11].
- IP5: Find point sources from an additional boundary observation [43–51]. Related inverse problems have been studied also in [52].

We note that in IP1, IP2, IP2a to identify  $f(x, t)$  or  $f(x)$  the solution  $u$  should be available in the whole physical domain  $\Omega$  that is hardly realized in practice. To overcome this deficiency, we now approach to the source inverse problem from another point of view: measure the solution  $u$  at some interior (or boundary) points  $x_1, x_2, \dots, x_N \in \Omega$  (or on  $\partial\Omega$ ) and from these data determine a term in the right hand side of (1.7). As any measurement is an average process, the following data are collected:

$$l_k u = \int_\Omega \omega_k(x)u(x, t)dx = h_k(t), \quad h_k \in L^2(0, T), \quad k = 1, 2, \dots, N, \quad (1.11)$$

with  $\omega_k \in L^\infty(\Omega)$  and  $\int_\Omega \omega_k(x)dx > 0$ ,  $k = 1, 2, \dots, N$ , being weight functions,  $N$  the number of measurements. We note that if we take

$$\omega_k(x) = \begin{cases} \frac{1}{|\Omega_k|}, & \text{if } x \in \Omega_k, \\ 0, & \text{otherwise,} \end{cases} \quad (1.12)$$

with  $|\Omega_k|$  being the volume of  $\Omega_k$ , a neighbourhood of  $x_k$ . Then  $l_k u$  shows the result of the measurement at  $x_k$  and can be understood as an average of  $u(x_k, t)$  if it exists. If we let  $|\Omega_k|$  tend to zero it will converge to  $u(x_k, t)$  if the last exists. However, since in general the solution  $u$  is understood in the weak sense,  $u(x_k, t)$  does not always make sense. Thus, the assumption that  $l_k u$  are available is meaningful in practice. Further, it is clear that if only  $l_k u$  are available, the uniqueness will not be guaranteed except for the case of determining  $f(t)$  in IP3, IP3a (see [26,27]). Hence, to avoid this ambiguity, assume that an a-priori information  $f^*$  of  $f$  is available which is reasonable in practice. In short, our inverse problem setting is as follows:

Suppose that  $l_k u = h_k(t)$ ,  $k = 1, 2, \dots, N$ , are available with some noise and an a-priori information  $f^*$  of  $f$  is available. Identify  $f$ .

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