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Preconditioning Toeplitz-plus-diagonal linear systems using the Sherman–Morrison–Woodbury formula

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ABSTRACT

In order to solve the Toeplitz-plus-diagonal linear systems arising from image restorations efficiently, we propose a sparse approximate inverse preconditioner based on the Sherman–Morrison–Woodbury formula. The preconditioner can be constructed through an incomplete factorization combined with some dropping strategies. When the preconditioner is applied to the conjugate gradient method for solving the Toeplitz-plus-diagonal linear systems, numerical results show that our preconditioning method is more effective than other existing ones.

1. Introduction

Toeplitz systems often appear in fields such as image processing, numerical analysis [1,2], etc. An n-by-n Toeplitz matrix takes the form

$$T_n = (t_{j-i})_{i,j=1}^{n} = \begin{pmatrix} t_0 & t_1 & \cdots & t_{n-2} & t_{n-1} \\ t_{-1} & t_0 & \cdots & t_{n-3} & t_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{-n+2} & t_{-n+3} & \cdots & t_0 & t_1 \\ t_{-n+1} & t_{-n+2} & \cdots & t_{-1} & t_0 \end{pmatrix}. \tag{1.1}$$

Specially, an m-by-m block Toeplitz matrix with n-by-n Toeplitz blocks taking the form

$$T_{mn} = \begin{pmatrix} T_0 & T_1 & \cdots & T_{m-2} & T_{m-1} \\ T_{-1} & T_0 & \cdots & T_{n-3} & T_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{-m+2} & T_{-n+3} & \cdots & T_0 & T_1 \\ T_{-m+1} & T_{-n+2} & \cdots & T_{-1} & T_0 \end{pmatrix}, \tag{1.2}$$

is called the BTTB matrix, where the blocks T_i ($i = 0, \pm 1, \ldots, \pm(m-1)$) are themselves Toeplitz matrices of order n. In this paper, we mainly focus on solving the Toeplitz-plus-diagonal linear systems of the form

$$(T_{mn} + D)x = b, \tag{1.3}$$

where T_{mn} is defined in (1.2), $D \in R^{mn\times mn}$ is a diagonal matrix and $b \in R^{mn}$ is the right vector.

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In order to implement the image restoration, a 2-dimensional deconvolution problem should be considered [3–7], which is an inverse process of 2-dimensional convolution. As for an N-by-N image X, the corresponding 2-dimensional convolution operation can be expressed in the matrix–vector notation as

$$b = Ax, \tag{1.4}$$

where the vector $x \in R^{N^2}$ is formed by row ordering the original image and contains the concatenated rows of the matrix X, and A is a $(2M + N - 2)$-by-N block column circulant matrix with $(2M + N - 2)$-by-N column circulant blocks as the following [7]

$$A = \begin{pmatrix} A_{-M+1} & & & 0 \\ A_{-M+2} & A_{-M+1} & & \\ \vdots & \ddots & \ddots & \\ A_0 & & \ddots & A_{-M+1} \\ \vdots & \ddots & & \vdots \\ A_{M-2} & & \ddots & \\ A_{M-1} & \ddots & & A_0 \\ & A_{M-1} & \ddots & \\ & & \ddots & \vdots \\ 0 & & & A_{M-1} \end{pmatrix}, \tag{1.5}$$

where the blocks A_i $(i = 0, \pm 1, \ldots, \pm(M-1))$ are themselves column circulant matrices of the form

$$A_i = \begin{pmatrix} a_{i,-M+1} & & & 0 \\ a_{i,-M+2} & a_{i,-M+1} & & \\ \vdots & \ddots & \ddots & \\ a_{i,0} & & \ddots & a_{i,-M+1} \\ \vdots & \ddots & & \vdots \\ a_{i,M-2} & & \ddots & \\ a_{i,M-1} & \ddots & & a_{i,0} \\ & a_{i,M-1} & \ddots & \\ & & \ddots & \vdots \\ 0 & & & a_{i,M-1} \end{pmatrix}. \tag{1.6}$$

Obviously, the resulting vector b is of length $(2M + N - 2)^2$.

In the image restoration, the goal of deconvolution is to compute x with A and b known. This problem is generally modeled as the linear squares problem [4–8]

$$\min_x \|b - Ax\|_2. \tag{1.7}$$

However, since all the realistic image processing procedures involve noises and deconvolution algorithms are very sensitive to noises, some form of regularization needs to be incorporated to stabilize the computations. Tikhonov regularization is verified to have good recovery qualities when it is used to restore blurred and noisy images in recent years [4–7]. Based on it, the original problem (1.7) is transformed into the following problem

$$\min_x \left\| \begin{pmatrix} b \\ 0 \end{pmatrix} - \begin{pmatrix} A \\ \lambda L \end{pmatrix} x \right\|_2, \tag{1.8}$$

where the scalar $\lambda \neq 0$ is a regularization parameter and the matrix $L \in R^{N^2 \times N^2}$ is a regularization operator. As is well known, the solution of (1.8) can be obtained by solving its normal equation

$$(\lambda^2 L^T L + A^T A)x = A^T b. \tag{1.9}$$

The regularization matrix L usually can be the identity matrix or a discrete form of the first or second derivative [4–7]. When L is the identity matrix I, the restoration model minimizes $\|x\|_2^2$, which can be viewed as the energy of the image X [1].