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Algorithms for solving the inverse problem associated with $KAK = A^{s+1}$

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ABSTRACT

In previous papers, the authors introduced and characterized a class of matrices called $\{K, s+1\}$-potent. Also, they established a method to construct these matrices. The purpose of this paper is to solve the associated inverse problem. Several algorithms are developed in order to find all involutory matrices K satisfying $KA^{s+1}K = A$ for a given matrix $A \in \mathbb{C}^{n\times n}$ and a given natural number s. The cases $s = 0$ and $s \geq 1$ are separately studied since they produce different situations. In addition, some examples are presented showing the numerical performance of the methods.

1. Introduction

Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, that is $K^2 = I_n$, where I_n denotes the identity matrix of size $n \times n$. In [1], the authors introduced and characterized a new kind of matrices called $\{K, s+1\}$-potent where K is involutory. We recall that for an involutory matrix $K \in \mathbb{C}^{n\times n}$ and $s \in \{0, 1, 2, \ldots\}$, a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent if $KA^{s+1}K = A$. These matrices generalize all the following classes of matrices: $\{s+1\}$-potent matrices, periodic matrices, idempotent matrices, involutory matrices, centrosymmetric matrices, mirrorsymmetric matrices, 2×2 circulant matrices, etc. Some related classes of matrices are studied in [2–8].

We emphasize that the role of centrosymmetric matrices is very important in different technical areas. We can mention among them antenna theory, pattern recognition, vibration in structures, electrical networks and quantum physics (see [9–15]). It is observed that the computational complexity of various algorithms is reduced taking advantage of the structure of these matrices. Also, mirror-symmetric matrices have important applications in studying odd/even-mode decomposition of symmetric multiconductor transmission lines [16]. Additional applications of $\{K, s+1\}$-potent matrices are related to the calculation of high powers of matrices, such as those needed in Markov chains and Graph Theory. Allowing negative values for s, Wikramaratna studied in [17] a new type of matrices for generating pseudo-random numbers. Inspired by this idea, another application, in image processing, has been considered in [18] where algorithms for image blurring/deblurring are designed. The advantage of this method is to avoid the computation of inverses of matrices and it can be applied, for instance, to protect a part of an image.

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The class of $\{K, s+1\}$-potent matrices is linked to other kind of matrices such as $\{s+1\}$-generalized projectors, $\{K\}$-Hermitian matrices, normal matrices, Hamiltonian matrices, etc. [19]. Moreover, some related results are given in [20] from an algebraic point of view. Furthermore, in [21] the authors developed an algorithm to construct the matrices in this class. This problem is called the direct problem.

The aim of this paper is to solve the inverse problem, that is, to find all the involutory matrices K for which a given matrix A is $\{K, s+1\}$-potent. For this purpose, several algorithms will be developed. The $s = 0$ and $s \geq 1$ cases are separately studied since they produce different situations. In addition, some examples are presented showing the numerical performance of the methods.

In what follows, we will need the spectral theorem:

Theorem 1 *([22]). Let $A \in \mathbb{C}^{n\times n}$ with k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then the matrix A is diagonalizable if and only if there are disjoint projectors $P_1, P_2, \ldots, P_k$, (i.e., $P_iP_j = \delta_{ij}P_i$ for $i, j \in \{1, 2, \ldots, k\}$) such that $A = \sum_{j=1}^{k} \lambda_j P_j$ and $I_n = \sum_{j=1}^{k} P_j$.*

For a positive integer k, let Ω_k be the set of all kth roots of unity. If we define $\omega = e^{2\pi i/k}$ then $\Omega_k = \{\omega^1, \omega^2, \ldots, \omega^k\}$. The elements of Ω_k will always be assumed to be listed in this order. Define $\Lambda_k = \{0\} \cup \Omega_k = \{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ so that $\lambda_0 = 0$, and $\lambda_j = \omega^j$ for $1 \leq j \leq k$.

Let $\mathbb{N}_s = \{0, 1, 2, \ldots, (s+1)^2 - 2\}$ for an integer $s \geq 1$. In [1], it was proved that the function $\varphi : \mathbb{N}_s \to \mathbb{N}_s$ defined by $\varphi(j) \equiv j(s+1)\ [\mathrm{mod}((s+1)^2 - 1)]$ is a permutation. Moreover, we notice that φ is an involution. It was also shown that the eigenvalues of a $\{K, s+1\}$-potent matrix A are included in the set $\Lambda_{(s+1)^2-1}$ and such a matrix A has associated certain projectors. Specifically, we will consider matrices P_j's satisfying the relations

$$KP_jK = P_{\varphi(j)} \quad \text{and} \quad KP_{(s+1)^2-1}K = P_{(s+1)^2-1}, \tag{1}$$

for $j \in \mathbb{N}_s$, where $P_0, P_1, \ldots, P_{(s+1)^2-1}$ are the eigenprojectors given in Theorem 1. For simplicity in the notation, we are assuming that all of these projectors P_j's are spectral projectors for A. However, for any specific spectral decomposition of A we must consider the (unique) spectral projectors needed for that decomposition. The designed algorithms compute these specific eigenprojectors and, in Section 5, we show some examples with their adequate spectral projectors. Those examples also illustrate all the studied situations throughout the paper.

Theorem 2 *([1]). Let $A \in \mathbb{C}^{n\times n}$ and $s \geq 1$ be an integer. Then the following conditions are equivalent:*

(a) *A is $\{K, s+1\}$-potent.*
(b) *A is diagonalizable, $\sigma(A) \subseteq \Lambda_{(s+1)^2-1}$, and the P_j's satisfy condition (1), where $\sigma(A)$ denotes the spectrum of A.*
(c) *$A^{(s+1)^2} = A$, and the P_j's satisfy condition (1).*

2. Obtaining the involutory matrices K for $s \geq 1$

It is well known that the Kronecker product is an important tool to solve some matrix problems, as for example the Sylvester and Lyapunov equations. The Kronecker sum, obtained as a sum of two Kronecker products, is applied, for example, to solve the two-dimensional heat equation, to rewrite the Jacobi iteration matrix, etc. [23]. The notation $\otimes$ used in this paper refers to the Kronecker product; and X^T denotes the transpose of the matrix X [24]. For any matrix $X = [x_{ij}] \in \mathbb{C}^{n\times n}$, let $v(X) = [v_k] \in \mathbb{C}^{n^2\times 1}$ be the vector formed by stacking the columns of X into a single column vector. The expression $[v(X)]_{\{(j-1)n+1,\ldots,(j-1)n+n\}}$, for $j = 1, \ldots, n$, denotes the jth column of X.

In what follows, we will need the following property: if $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ then

$$\mathrm{Ker}(A) \cap \mathrm{Ker}(B) = \mathrm{Ker}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right), \tag{2}$$

which is also valid for a finite number of matrices of suitable sizes, where Ker(.) denotes the null space of the matrix (.).

We recall that when A is a diagonalizable matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_l$, the principal idempotents are given by

$$P_t = \frac{p_t(A)}{p_t(\lambda_t)} \quad \text{where } p_t(\eta) = \prod_{\substack{i=1 \\ i\neq t}}^{l} (\eta - \lambda_i). \tag{3}$$

By using the function φ and the projectors introduced in (1), it is possible to construct the matrix

$$M = \begin{bmatrix} (P_0^T \otimes I_n) + (I_n \otimes -P_{\varphi(0)}) \\ (P_1^T \otimes I_n) + (I_n \otimes -P_{\varphi(1)}) \\ \vdots \\ (P_{(s+1)^2-2}^T \otimes I_n) + (I_n \otimes -P_{\varphi((s+1)^2-2)}) \\ (P_{(s+1)^2-1}^T \otimes I_n) + (I_n \otimes -P_{(s+1)^2-1}) \end{bmatrix}. \tag{4}$$