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A front-fixing numerical method for a free boundary nonlinear diffusion logistic population model

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The spatial–temporal spreading of a new invasive species in a habitat has interest in ecology and is modeled by a moving boundary diffusion logistic partial differential problem, where the moving boundary represents the unknown expanding front of the species. In this paper a front-fixing approach is applied in order to transform the original moving boundary problem into a fixed boundary one. A finite difference method preserving qualitative properties of the theoretical solution is proposed. Results are illustrated with numerical experiments.

1. Introduction

For the sake of clarity in the terminology and as many authors undistinguish the terms free boundary and moving boundary, we recall these concepts following Crank approach [1]. A moving boundary problem is characterized by the fact that the boundary of the domain is not known in advance but it has to be determined as a part of the solution. These problems are often called Stefan problems due to the Stefan condition that links the behavior of the boundary with the unknown solution, see [1,2]. The term free-boundary problem is commonly used when the boundary is independent of the time and typically related to elliptic problems. Moving boundary problems have their origins in physical and engineering problems [1,3], and more recently in biological and physiological sciences [4], decision and control theory and ecology [5].

Prior to [5] the modeling of biological invasions has been widely studied in [6–14] under the crucial restriction that in the previous papers the spatial domain is not constrained by the population behavior, that is the essence of the Stefan condition. The first diffusive logistic model related to biological invasions was initiated in 1937, of course without boundary restrictions, independently by Fisher [6] and Kolmogorov–Petrovsky–Piskunov (KPP) [7]. Very recent papers have treated numerically these nonlinear models focusing on the stability and the preservation of the qualitative properties of the theoretical solution [15–17].

To our knowledge the seminal paper [5] by Du and Lin is the first contribution in the field of spreading of populations where a Stefan condition is used and managing a moving boundary problem of parabolic type. Further developments of this problem have been treated in [18–20]. The diffusive logistic model of [5] for the density of population of the invasive species $U(t,x)$ depending on time t and spatial variable x states as follows:

$$\frac{\partial U}{\partial t} - D\frac{\partial^2 U}{\partial x^2} = U(a - bU), \quad t > 0,\ 0 < x < H(t), \tag{1}$$

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together with the boundary conditions

$$\frac{\partial U}{\partial x}(t,0)=0, \qquad U(t,H(t))=0, \quad t>0, \tag{2}$$

the Stefan condition

$$H'(t)=-\mu\frac{\partial U}{\partial x}(t,H(t)), \quad t>0, \tag{3}$$

and the initial conditions

$$H(0)=H_0, \qquad U(0,x)=U_0(x), \quad 0\le x\le H_0. \tag{4}$$

The initial function $U_0(x)$ satisfies the following properties:

$$U_0(x)\in C^2([0,H_0]), \qquad U_0'(0)=U_0(H_0)=0, \qquad U_0(x)>0, \quad 0\le x<H_0. \tag{5}$$

Here $H(t)$ is the unknown moving boundary such that the population is distributed in the interval $[0,H(t)]$, $D>0$ is the dispersal rate and the positive parameters a and b are the intrinsic growth rate and the intraspecific competition, respectively. The parameter $\mu>0$ involved in the Stefan condition (3) is the proportionality constant between the population gradient at the front and the speed of the moving boundary. Unlike to the previous models, where only spreading behavior was admissible, the authors of [5] show by the very first time a dichotomic alternative behavior, vanishing or spreading approach to the habitat carrying capacity a/b, depending on the initial front and population density and the value of the parameter μ appearing in Stefan condition. According to [5] there is a threshold value μ^* whose value is not known in advance splitting the vanishing–spreading behavior.

This paper aims to be a continuation and numerical completion of [5] with the conviction that the best model may be wasted with a careless numerical treatment. Apart from the computation of the population density solution of problem (1)–(4) and the numerical analysis detailed below, this paper has the potential advantage that allows us the computation of the expanding front of the species population as well as the approximation of the crucial parameter μ^* whose existence is guaranteed in [5], but whose value is not known in terms of data problem. A brief numerical treatment of the problem may be found in Section 3.6 of [21].

This paper is organized as follows. In Section 2, and following the trajectory of the authors in the study of finance problems (see [22,23]), we use the well-known Landau transformation (see [1,24]), in order to convert the problem (1)–(4) into a fixed spatial domain one, where the moving boundary is included as another variable to solve apart from the population density. We also include in Section 2 the discretization of the transformed problem achieving an explicit finite difference scheme allowing the computation not only of the population but also of the expanding front. Section 3 deals with the study of the consistency of the scheme with the transformed problem. Dealing with population problems it is important to guarantee the positivity of the numerical solution; this qualitative property together with the stability of the numerical solution and the positivity and monotone behavior of the numerical expanding front are studied in Section 4. Section 5 illustrates with numerical examples the dichotomic behavior of the numerical solution of the problem.

2. Transformation and discretization of the continuous problem

Let us begin this section by transforming the moving front problem (1)–(4) into a problem with a fixed domain [0, 1]. Let us consider the Landau transformation, [1,24],

$$z(t,x)=\frac{x}{H(t)}, \quad W(t,z)=U(t,x). \tag{6}$$

Under substitution (6) problem (1)–(4) takes the form:

$$G(t)\frac{\partial W}{\partial t}-G'(t)\frac{z}{2}\frac{\partial W}{\partial z}-D\frac{\partial^2 W}{\partial z^2}=G(t)W(a-bW), \quad t>0,\ 0<z<1, \tag{7}$$

where:

$$G(t)=H^2(t), \quad t\ge 0. \tag{8}$$

Boundary conditions (2) and Stefan condition (3) take the form:

$$\frac{\partial W}{\partial z}(t,0)=0, \qquad W(t,1)=0, \quad t>0, \tag{9}$$

and

$$G'(t)=-2\mu\frac{\partial W}{\partial z}(t,1), \quad t>0, \tag{10}$$