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# Compressed sensing image restoration based on data-driven multi-scale tight frame☆

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ABSTRACT

It has been shown that redundant signal representation, e.g. tight frame, plays important role in compressed sensing image restoration. In order to get a good sparse representation, one has made enduring efforts to pursue tight frames. Although there are some tight frames under which a type of images has a good sparse approximation, another type of images may not have sparse approximation because of the images' great difference in structure. This paper presents a novel compressed sensing image restoration method based on data-driven multi-scale tight frame. This method derives a discrete multi-scale tight frame system adaptive to the original image from the input compressed sensing image. Such an adaptive tight frame construction scheme is applied to compressed sensing image restoration. The experimental results show our approach's efficiency.



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## 1. Introduction

Compressed Sensing [1,2] (CS) concerns the problem of recovering a high-dimensional sparse signal $f$ from a small number of linear measurements

$$g = Af + e, \tag{1}$$

where $A$ is an $m \times n$ sensing matrix with $m \ll n$ independent of signal $f$ such as Gaussian sensing matrices, and $e \in R^m$ is a noise term modeling measurement error. The goal is to restore the unknown signal $f \in R^n$ based on the measurements

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$g \in R^m$. Generally, the unknown signal $f$ is assumed to be sparse or compressible in an orthogonal basis in traditional CS literature. In this paper, signals are assumed to be sparse in a redundant frame $D \in R^{n\times d}(d > n)$, that is $f = Dx$, where $x$ is sparse or compressible in the frame $D$.

One effective method of restoring the original $f$ from the measurements $g$ is given by the $\ell_1$-analysis method. Empirical studies have shown very promising results for the $\ell_1$-analysis problem [3]:

$$\hat{f} = \arg\min_{\hat{f}\in R^n} \left\| D^*\hat{f} \right\|_1 \quad \text{s.t.} \quad \left\| g - A\hat{f} \right\|_2 \le \varepsilon \tag{2}$$

where $\varepsilon$ is a likely upper bound on the noise level. It was shown that when $D$ is an $n \times d$ arbitrary tight frame and the measurement matrix $A$ is an $m \times n$ independent identically distributed (i.i.d.) Gaussian matrix with $m$ on the order of $s\log(d/s)$, the solution to (2) satisfied the following bound

$$\left\| f - \hat{f} \right\|_2 \le C_0\varepsilon + C_1 \frac{\left\| D^*f - (D^*\hat{f})_s \right\|_1}{\sqrt{s}}, \tag{3}$$

for some numerical constants $C_0$ and $C_1$, and where $(D^*f)_s$ is the vector consisting of the largest $s$ entries of $D^*f$ in magnitudes.

Tight frames have received increased attention in image processing applications in order to efficiently represent natural images, including ridgelets, shearlets, curvelet, and so on [4–8]. However, this fixed redundant signal presentation system may be applicable to cartoon-type images but not to texture-type images, for their heavily dependence on some functional assumptions only under which natural images are sparse or compressible. There are various natural images varying greatly in terms of geometrical structure. The tight frame constructed for efficiently sparsilizing one type of images may not always sparsilize the other type of ones [9,10]. Thus, developing a tight frame system driven by the input data with better performance in terms of sparse approximation is a better method.

In recent years, "adaptivity" has been increased concerned by learning approaches [11–14], which get better sparsity by learning an over complete dictionary from the input signals, especially for the natural textural images. As a result, these learning approaches tend to be outstanding of the image restoration methods based on the signal's sparsity. However, the learned over complete dictionaries lack several "good" properties which is desired for many applications in image restoration. The perfect reconstruction property is in the first place, for which ensures that the input signals can be represented by its canonical expansion perfectly like the orthonormal bases [9]. Also, it is still challenging to develop a stable numerical algorithm for evaluating the optimal over complete system.

According to above reasons, we aim to develop a novel approach to achieve a discrete data-driven multi-scale tight frame system in this paper. The constructed tight frame would like to approximate the input image more sparsely than the existing fixed tight frames. In contrast to the existing fixed frame (FF) [15] and the general learned over complete dictionaries (e.g. the K-SVD method [16]), the constructed multi-scale tight frames can efficiently capture the complex texture patterns in the compressed sensing image restoration experiments, and have the property of "perfect reconstruction". Moreover, the minimization problems arising in the construction of multi-scale tight frames are better conditioned than those of general learned over-complete dictionaries, owing to the Parseval identity for tight frames. Numerical experiments show that the constructed tight system enhances the recovery performance compared with the FF and the K-SVD method.

## 2. Preliminaries

We first present here some preliminaries about frames and tight frames in a Hilbert space $H$, and then give the method of generating multi-level frame operator from the frame filters. Let $Z$ denote the set of all integers, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the usual inner product, a norm of a Hilbert space $H$, respectively. If let a vector $v = (v_1, v_2, \ldots, v_d)^T$, then we denote $\|v\|_p = \left(\sum_{i=1}^d |v_i|^p\right)^{1/p}$, $0 < p < \infty$. By abuse of notation, $\|v\|_0 := \#\{i : v_i \neq 0\}$. If $\|v\|_0 \le k$, we say that the vector $v$ is $k$-sparse. By a sparse representation, we mean that for a signal of length $d$, we can represent it with $k \ll d$ nonzero coefficients; by a compressible representation, we mean that the signal is well-approximated by a signal with only $d$ nonzero coefficients.

A sequence $\{\varphi_n\}_{n\in Z} \subset H$ is called a frame for $H$ if there are two positive constant numbers $c_1$ and $c_2$ such that

$$c_1 \|f\|_2^2 \le \sum_{n\in Z} |\langle\varphi_n, f\rangle|^2 \le c_2 \|f\|_2^2, \quad \forall f \in H \tag{4}$$

when $c_1 = c_2 = 1$, the frame $\{\varphi_n\}_{n\in Z}$ is a tight frame for $H$. There are two operators associated with a given frame $\{\varphi_n\}_{n\in Z}$: the analysis operator $D$ defined by [9,10]

$$D : f \in H \to \{\langle f, \varphi_n\rangle\} \in \ell^2(Z), \tag{5}$$

and its adjoint operator $D^*$, also called the synthesis operator, defined by

$$D^* : \{a_n\} \in \ell^2(Z) \to \sum_n a_n\varphi_n \in H. \tag{6}$$