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The method of steepest descent for estimating quadrature errors

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ABSTRACT

This work presents an application of the method of steepest descent to estimate quadrature errors. The method is used to provide a unified approach to estimating the truncation errors which occur when Gauss–Legendre quadrature is used to evaluate the nearly singular integrals that arise as part of the two dimensional boundary element method. The integrals considered here are of the form $\int_{-1}^{1} \frac{h(x)\,dx}{((x-a)^2+b^2)^\alpha}$, where $h(x)$ is a "well-behaved" function, $\alpha > 0$ and $-1 < a < 1$. Since $0 < b \ll 1$, the integral is "nearly singular", with a sharply peaked integrand.

The method of steepest descent is used to estimate the (generally large) truncation errors that occur when Gauss–Legendre quadrature is used to evaluate these integrals, as well as to estimate the (much lower) errors that occur when Gauss–Legendre quadrature is performed on such integrals after a "sinh" transformation has been applied. The new error estimates are highly accurate in the case of the transformed integral and are shown to be comparable to those found in previous work by Elliott and Johnston (2007). One advantage of the new estimates is that they are given by just one formula each for the un-transformed and the transformed integrals, rather than the much larger set of formulae in the previous work. Another advantage is that the new method applies over a much larger range of α values than the previous method.

1. Introduction

1.1. Background

The boundary element method [1] is a standard technique that is used to solve many important engineering problems and, when it is implemented, it is necessary to evaluate numerically three main types of integrals: non-singular, singular and nearly singular. We focus here on the integrals that occur when the source point is very near to, but not on, the element of integration and which are described as "nearly singular". In this case the integrand develops a sharp peak, with the result that Gauss–Legendre quadrature is unable to produce good approximations to the integral.

The accurate evaluation of these integrals is particularly important when calculations must be made near to the boundary; for example, when calculating the solution in potential problems or evaluating stresses and displacements

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in plane elasticity problems. Examples of such problems include studies of:- shell-like structures [2], transient heat conduction [3], elasticity [4], elastoplastic contact [5], stress intensity sensitivities [6], crack growth [7] and thin layered coatings [8].

A number of different methods have been proposed to evaluate the nearly singular integrals that occur in a 2D application of the boundary element method. One group of such methods involves the use of non-linear transformations, such as the cubic polynomial transformation [9,10], the bi-cubic transformation [11], the sigmoidal transformation [12], the co-ordinate optimisation transformation [13] and the distance transformation [14]. More recently introduced transformations include the exponential transformation [15] and the very accurate and easily implemented so-called sinh transformation method [16–18], which has been shown to be more accurate, by several orders of magnitude [16], than simply using Gauss–Legendre quadrature. The advantages and disadvantages of these and other approaches are well discussed in the paper by Zhang and Sun [15].

In Section 2 we will use the method of steepest descent to provide estimates of the error that occurs when Gauss–Legendre quadrature is used to evaluate 1D nearly singular integrals of the form given in Section 1.2. Then, in Section 3, we adopt a similar method of steepest descent approach, this time applied to integrals that are first transformed by a sinh transformation after which Gauss–Legendre quadrature is applied. Both types of error estimates will then be compared in Section 4 to the exact errors, as well as to previous error estimates that are produced by a different method [17] and then final conclusions will be drawn.

1.2. Quadrature error

In this paper we propose to use the method of steepest descent in order to estimate the truncation error when n-point Gauss–Legendre quadrature is used to evaluate the integral $I(\alpha, a, b)$ defined by

$$I(\alpha, a, b) := \int_{-1}^{1} \frac{h(x)\,dx}{((x-a)^2+b^2)^{\alpha}}, \tag{1.1}$$

where $h(x)$ is a real function which is "well-behaved". Here we will assume that $\alpha > 0$, $-1 < a < 1$ and $0 < b \ll 1$, so that the integrand has a singularity close to the interval of integration $(-1, 1)$.

To be precise, in this work we consider the integrals

$$I_k(\alpha, a, b) := \int_{-1}^{1} \frac{h_k(x)\,dx}{((x-a)^2+b^2)^{\alpha}}, \tag{1.2}$$

where $h_1(x) \equiv 1$, $h_2(x) = x - a$, $h_3(x) = 1 - x^2$ and $h_4(x) = \sqrt{1+x^2}$, since these are examples of integrals of the type that appear in the 2D boundary element method [16,17]. Here $h_k(x)$ is either a real polynomial (basis function) that does not have zeros at the same points as the denominator of (1.1) (I_1, I_2 and I_3) or a function that has arisen from the Jacobian of the transformation (I_4).

From [19, §25.4.29] we have, for n-point Gauss–Legendre quadrature that

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_{i,n} f(x_{i,n}) + E_n(f), \tag{1.3}$$

where $x_{i,n}$ are the zeros of the Legendre polynomial P_n and the weights $w_{i,n}$ are given by $2/((1-x_{i,n}^2)(P_n'(x_{i,n}))^2)$. On assuming that the definition of the integrand f can be continued into the complex plane, Donaldson and Elliott [20] have shown that the truncation error $E_n(f)$ can be expressed as a contour integral given by

$$E_n(f) = \frac{1}{2\pi i} \int_{\mathcal{C}} k_n(z) f(z)\,dz, \tag{1.4}$$

where $\mathcal{C}$ is a closed contour enclosing the interval $(-1, 1)$, described positively (i.e. counterclockwise), and such that f is analytic on and within $\mathcal{C}$. The function k_n is defined by

$$k_n(z) := \frac{1}{P_n(z)} \int_{-1}^{1} \frac{P_n(t)\,dt}{z-t}, \quad z \notin [-1, 1]. \tag{1.5}$$

On assuming that $n \gg 1$ we have, for z bounded away from the interval $[-1, 1]$, that

$$k_n(z) = \frac{c_n}{(z+\sqrt{z^2-1})^{2n+1}}, \tag{1.6}$$

where

$$c_n := 2\pi \frac{(\Gamma(n+1))^2}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}, \tag{1.7}$$

which is approximately 2π for $n \gg 1$.