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New fully symmetric and rotationally symmetric cubature rules on the triangle using minimal orthonormal bases

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ABSTRACT

Cubature rules on the triangle have been extensively studied, as they are of great practical interest in numerical analysis. In most cases, the process by which new rules are obtained does not preclude the existence of similar rules with better characteristics. There is therefore clear interest in searching for better cubature rules.

Here we present a number of new cubature rules on the triangle, exhibiting full or rotational symmetry, that improve on those available in the literature either in terms of number of points or in terms of quality. These rules were obtained by determining and implementing minimal orthonormal polynomial bases that can express the symmetries of the cubature rules. As shown in specific benchmark examples, this results in significantly better performance of the employed algorithm.

1. Introduction

Cubature, that is the numerical computation of a multiple integral, is an important method of numerical analysis, as it is of great practical interest in different applications involving integration. An extensive literature therefore exists on this topic (see e.g. [1,2]), including also compilations of specific cubature rules [3].

The present paper considers cubature rules on the triangle. This is perhaps the most studied cubature domain, with a correspondingly large body of literature a selection of which is presented here. While rules of degree up to 20, thus covering most cases of practical interest, were progressively developed by 1985 [1,4–6], this is still an active field [7–16]. This activity persists for two distinct reasons, the first being that different applications require different properties of the cubature rules; the previously cited work for example focuses only on fully symmetric rules (which are also the easier to determine), while only a few works consider rotationally symmetric [17,12] or asymmetric [18,19] rules. The second reason explaining the interest in researching new cubature rules is that almost all rules in the literature have been determined numerically using an iterative procedure, therefore there is the possibility that a "better" rule (matching some given requirements) may exist, for example one having fewer points (see [20] for a lower bound on the number of points for given degree). For fully symmetric rules, the fact that the "best" existing rules for degree up to 14 have indeed the minimal possible number of points was recently proved using solutions based on algebraic solving [15].

In this paper we focus on the iterative algorithm for obtaining fully symmetric cubature rules on the triangle initially proposed by Zhang et al. [11] and recently refined by Witherden and Vincent [16]. A main feature of [16] (which had already been used in [18,12]) is the use of an orthonormal basis instead of the monomial basis usually employed. Further improving upon this point, we describe here a minimal orthonormal basis for fully symmetric rules and then extend this basis to also

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cover the case of rules with only rotational symmetry. This results in a number of new rules that improve upon those found in the literature, especially for the rotationally symmetric case.

The structure of the paper is as follows: after this introduction, Section 2 summarises the required theoretical background. Section 3 presents orthonormal bases for the fully symmetric case, including minimal bases, in terms of the typically used orthonormal polynomials while Section 4 presents the minimal basis in terms of symmetric polynomials. In Section 5 we extend the minimal basis to obtain a minimal basis for cubature rules with rotational symmetry. A summary of the numerical results is presented and discussed in Section 6, while the conclusions of the paper are stated in Section 7.

2. Theoretical background

A cubature rule approximates the integral of a function f on a domain Ω (normalised by the domain's area A) as the weighted sum of the function's value evaluated at a set of n_k points $\boldsymbol{x}_i$,

$$\sum_i^{n_k} w_i f(\boldsymbol{x}_i) \approx \frac{1}{A}\int_\Omega f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{1}$$

The cubature rule is of polynomial degree ϕ if Eq. (1) is exact for all polynomials of degree up to ϕ but not exact for at least one polynomial of degree $\phi+1$.

Since Eq. (1) is linear in the function f, we only need to ensure that it is exact for a basis of the polynomials of degree ϕ. The simplest such basis in two dimensions is the set of monomials $x^i y^j$ in the Cartesian coordinates x and y with $i+j \le \phi$, but for the triangle another simple basis is the set of monomials $L_1^i L_2^j L_3^{\phi-i-j}$ expressed in term of the areal (or barycentric) coordinates L_1, L_2 and L_3 (with all exponents being non-negative).

In two dimensions each point contributes three unknowns (two coordinates and a weight), therefore setting in Eq. (1) f as each of the basis polynomials for degree ϕ results in a polynomial system of $(\phi+1)(\phi+2)/2$ equations in $3n_k$ variables. The solution of this system yields the cubature point coordinates and weights defining the cubature rule.

In the general (asymmetric) case it can be quite difficult to solve the above-mentioned system even for moderate values of ϕ, therefore some symmetry conditions are imposed on the cubature points to reduce the number of unknowns. As mentioned in the introduction, these symmetries may also be a requirement of the application being considered; on the triangle, for example, full symmetry ensures that the computed approximate value of the integral is independent of the order in which the vertices are numbered.

For cubature rules on the triangle, the most commonly used symmetry is full symmetry, where if a point with areal coordinates (L_1, L_2, L_3) appears in the rule, then all points resulting from permutation of the areal coordinates also appear. Depending on the number of distinct values of the areal coordinates we therefore obtain different symmetry orbits (for 1, 2 or 3 distinct values we get orbits of type 0, 1 or 2 which have 1, 3 or 6 points and contribute 1, 2 or 3 unknowns to the system of equations); see [16] for a more detailed explanation of symmetry orbits. Full symmetry allows for a significant reduction in the number of unknowns (roughly by a factor of 6 for larger values of ϕ) and, through appropriate considerations, for a corresponding decrease in the number of equations [5].

The disadvantage of full symmetry is that in most cases it does not lead to the cubature rule with the minimal number of points for a given degree and quality (in the sense of "quality" defined in Section 6). It is possible to get rules with fewer points, while still reducing the number of equations and unknowns, by requiring only rotational symmetry. In this case instead of considering all the permutations of the areal coordinates we only consider the even permutations, so that (L_1, L_2, L_3) is permuted into (L_2, L_3, L_1) and (L_3, L_1, L_2). This results in two types of orbits: type-0 with only one point (the centroid) and type-1 with three points, therefore the number of unknowns is approximately twice that of the fully symmetric case.

3. Orthonormal bases on the triangle

3.1. A full orthonormal basis

While the monomials (in either the Cartesian or the areal coordinates) described in Section 2 are the simplest basis polynomials, they lead at higher degrees to polynomial systems which are poorly conditioned, therefore the use of an orthonormal basis has been proposed [18,12,16].

A standard set of orthonormal basis polynomials on the triangle has been proposed in the literature [21–23], which we can write in the form

$$\psi_{ij}(\boldsymbol{x}) = \hat{P}_i\big(d/s\big)\hat{P}_j^{(2i+1,0)}\big(1-2s\big)s^i \tag{2}$$

where $\hat{P}_n^{(\alpha,\beta)} = \sqrt{2n+\alpha+1}P_n^{(\alpha,\beta)}$ are scaled Jacobi polynomials and the values d and s depend on the coordinates.

Specific expressions for the $\psi_{ij}(\boldsymbol{x})$ (and therefore for d and s in Cartesian coordinates) are given in the literature by specifying a reference triangle. Using areal coordinates, however, d and s are simply the difference and sum respectively of two of the areal coordinates, without reference to a specific triangle. Choosing for example L_2 and L_1 we have

$$s = L_2 + L_1, \qquad d = L_2 - L_1. \tag{3}$$