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Computing Cauchy principal value integrals using a standard adaptive quadrature

Paweł Keller *, Iwona Wróbel

Warsaw University of Technology, Faculty of Mathematics and Information Science, ul. Koszykowa 75, 00-662 Warszawa, Poland

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ABSTRACT

We investigate the possibility of fast, accurate and reliable computation of the Cauchy principal value integrals $⨍_a^b f(x)(x-\tau)^{-1}dx$ $(a < \tau < b)$ using a standard adaptive quadrature. In order to properly control the error tolerance for the adaptive quadrature and to obtain a reliable estimation of the approximation error, we research the possible influence of round-off errors on the computed result. As the numerical experiments confirm, the proposed method can successfully compete with other algorithms for computing such type integrals. Moreover, the presented method is very easy to implement on any system equipped with a reliable adaptive integration subroutine.

1. Introduction

We consider the problem of numerical evaluation of the Cauchy principal value integral

$$I_{a,b,\tau}(f) := ⨍_a^b \frac{f(x)}{x-\tau}\,dx = \lim_{\mu\to 0^+}\left(\int_a^{\tau-\mu} + \int_{\tau+\mu}^b\right)\frac{f(x)}{x-\tau}\,dx, \tag{1.1}$$

where $\tau \in (a, b)$, and the function f has bounded first derivative. In general, the integral (1.1) exists if f is Hölder continuous (cf. [1, Section 1.6]). The integrals of this type appear in many practical problems related to aerodynamics, wave propagation or fluid and fracture mechanics, mostly with relation to solving singular integral equations.

A great many papers devoted to the problem of numerical evaluation of the integrals of the form (1.1) have been published so far. Some of them are [2–7]. A nice survey on the subject, along with a large number of references, is presented in [1, Section 2.12.8].

Even though so many algorithms have been known for a quite long time, the subroutines for computing the integrals of the type (1.1) are not commonly available in systems for scientific computations. This is probably because most of the methods assume some properties of the function f, e.g., that f is analytic, f is smooth enough, etc. On the other hand, almost every system for scientific computations is equipped with one or more subroutines for automatic computation of the integrals

$$\int_a^b \psi(x)\,dx. \tag{1.2}$$

* Corresponding author.
E-mail addresses: Pawel.Keller@mini.pw.edu.pl (P. Keller), Iwona.Wrobel@mini.pw.edu.pl (I. Wróbel).

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These algorithms, usually called *adaptive quadratures*, can compute the integrals of the form (1.2) for a very wide range of integrands.

The natural question arises: can an adaptive quadrature be successfully used to numerically compute the integral (1.1)? In this paper, we search for the positive answer to that question. Our goal is to provide tools that will allow to use an existing adaptive quadrature to compute an accurate approximation to the integral (1.1), and also will help to obtain a reliable error estimation of the computed result. A usual automatic (adaptive) quadrature performs the computations until the error tolerance provided by a user (or the default, fixed one) is met. The numerical value of the Cauchy principal value integral is sometimes quite sensitive to the influence of round-off errors. If a user wishes high accuracy of the approximation, it may be difficult to select a proper error tolerance (if it is set too small, then the computation time may considerably increase or even the quadrature may fail to deliver reliable approximation). Thus, in the method proposed in this paper, the error tolerance is trimmed at the safe level. In particular, if the tolerance is set to zero, the algorithm is expected to compute the best (or rather near the best, within reasonable limits) possible approximation to the integral (1.1) for given parameters f and τ.

The next section contains the formulation of, what may be called, the basic form of our algorithm. We apply two known analytical transformations that convert the integral (1.1) into the sum of two non-singular integrals. In Section 3, we briefly describe the idea of adaptive quadratures. In the subsequent section, we discuss the problem of possible loss of significant digits and the influence of round-off errors on the approximation to a Cauchy principal value integral computed using the formula derived in Section 2. The obtained error estimates are crucial for the efficiency and reliability of the proposed method. As we briefly demonstrate, these estimates can be used to increase reliability of other algorithms for computing integrals of the form (1.1). In Section 5, we present many numerical examples to validate the usefulness of the proposed algorithm. The method is thoroughly tested with three different adaptive quadratures: the built-in adaptive quadrature of the *Maple* system, the one included in *Matlab*, and the one presented in [8].

2. Analytical transformations

Without loss of generality, we may restrict our attention to the case $a = -1$ and $b = 1$. The computation of the Cauchy principal value integral

$$I_\tau(f) \equiv I_{\tau,-1,1}(f) = ⨍_{-1}^{1} \frac{f(x)}{x-\tau}\,dx \tag{2.1}$$

may at first seem quite easy if we observe that by a simple change of variables, setting

$$\delta = \min\{1+\tau, 1-\tau\}, \tag{2.2}$$

we obtain

$$\begin{aligned} I_\tau(f) &= \int_{|x-\tau|\ge\delta} \frac{f(x)}{x-\tau}\,dx + ⨍_{\tau-\delta}^{\tau+\delta} \frac{f(x)}{x-\tau}\,dx \\ &= \int_{|x-\tau|\ge\delta} \frac{f(x)}{x-\tau}\,dx + \int_0^{\delta} \frac{f(\tau+x)-f(\tau-x)}{x}\,dx, \end{aligned} \tag{2.3}$$

where we use the convention that $\int_{|x-\tau|\ge\delta} \equiv \int_{\tau+\delta}^{1}$ if $\delta = 1+\tau$, and $\int_{|x-\tau|\ge\delta} \equiv \int_{-1}^{\tau-\delta}$ if $\delta = 1-\tau$.

The formula (2.3) was applied for the first time by Longman in [4], and it was derived by splitting the function f into the odd and even parts. Both integrals on the right hand side of (2.3) exist in the Riemann sense. We should note that if the function f has bounded first derivative in the neighbourhood of τ, then the second integral is not even singular (unless f has singularities itself).

The first integral on the right hand side of (2.3) was commonly not paid attention to, as it is always a proper one. However, if $|\tau|$ is close to 1, then this integral is a near-singular one, and standard quadratures may fail when applied directly.

In many algorithms, another transformation of the integral $I_\tau(f)$ is used, usually called *subtracting out the singularity*. We have

$$\begin{aligned} I_\tau(f) &= ⨍_{-1}^{1} \frac{f(\tau)}{x-\tau}\,dx + \int_{-1}^{1} \frac{f(x)-f(\tau)}{x-\tau}\,dx \\ &= f(\tau)\,\log\frac{1-\tau}{1+\tau} + \int_{-1}^{1} \frac{f(x)-f(\tau)}{x-\tau}\,dx. \end{aligned} \tag{2.4}$$

A direct application of the above formula is commonly not recommended (see, e.g., [3,9]) due to possible severe cancellation if a quadrature node happens to be very close to τ.