



## Extinction-time for stochastic population models



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### ABSTRACT

The analysis of interacting population models is the subject of much interest in mathematical ecology. Moreover, the persistence and extinction of these models is one of the most interesting and important topics, because it provides insight into their behavior. The mean extinction-time for stochastic population models considered in this paper depends on the initial population size and satisfies a stationary partial differential equation, related to the backward Kolmogorov differential equation, a linear second-order partial differential equation with variable coefficients. In this communication we review several papers where we have proposed some numerical techniques in order to estimate the mean extinction-time for stochastic population models. Besides, we will compare the theoretical predictions and numerical simulations for stochastic differential equations (SDEs). This work can be viewed as a unified review of the contributions de la Hoz and Vadillo (2012), de la Hoz et al. (2014) and Doubova and Vadillo (2014).

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### 1. Introduction

Stochastic population models are systems of stochastic differential equations (SDE) with a deterministic part plus a stochastic one. In classic books such as for example [1] or [2], we can find results on the evolution, persistence and extinction for deterministic population models, but the stochastic models are more complicated and have quite different persistence and extinction behaviors.

The SDE system for the dynamics of  $n$  interacting populations has the following form:

$$\begin{cases} d\mathbf{X}(t) = \mu(t, \mathbf{X})dt + B(t, \mathbf{X})d\mathbf{W}(t), \\ \mathbf{X}(0) = \mathbf{X}_0, \end{cases} \quad (1)$$

where  $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))^t$  is an  $n$ -dimensional random variable, and  $\mathbf{W}(t) = (W_1(t), \dots, W_m(t))^t$  are  $m$  independent Wiener processes. The vectorial function  $\mu(t, \mathbf{X})$  is called the drift, and  $B(t, \mathbf{X})$  is the diffusion matrix.

Notice that if the matrix  $B \equiv 0$ , then this system reduces to a standard deterministic model for the population dynamics.

One of the major goals of stochastic population dynamics (1) is to predict finite persistence time, i.e. the time when one or more components of the population go to zero. In this paper,  $T$  will be the random variable that indicates the persistence time:

$$T := \inf\{t \geq 0 : X_i(t) = 0 \text{ for any } i = 1, \dots, n\}.$$

Notice that there are other definitions of persistence, as for example weak persistence, weak persistence in the mean, strong persistence in the mean, etc. (see [3, p. 444]). The stochastic variable  $T$  obviously depends on the initial population size  $\mathbf{X}_0$ .

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On the other hand, let us also introduce the expectation time until extinction

$$\tau := \mathbb{E}(T).$$

It is clear that the variable  $\tau$  also depends on the initial population size, i.e.  $\tau = \tau(\mathbf{X}_0)$ . Let us notice that this mean persistence time can be computed by solving stationary partial differential equations, which are related to the backward Kolmogorov equation (see [4, p. 443]; we refer to [5, p. 150] or Chapter 8 of [6] for a clear review of these equations). The backward Kolmogorov equations are linear second-order partial differential equations (PDEs) with variable coefficients. It is well known that it is possible to compute an analytical solution only in a few simple cases, see for example [4], for a population model with linear birth and death rates, or [7], where the exact expressions are given for the extinction-time of a class of birth–death processes. In general, we can only compute numerical approximations, see for example [8], where central difference approximations are used. In this paper we use a Finite Element Method (FEM). There is an ample bibliography about FEM, see for instance [9] or [10] for a good introduction. The more popular reference on FEM in solid mechanics are the books [11,12]. Generalization to other fields of physics or engineering have been done by applied mathematicians through the concept of variational formulation of partial differential equations (see for example [13]).

This paper is organized as follows. In Section 2, we describe a Lotka–Volterra predator–prey model. We will estimate its mean extinction-time using the numerical approximations of the associated Kolmogorov equation and we will make a direct comparison between predictions and numerical simulations. In Sections 3 and 4 we will analyze the mean extinction-time for more complicated models. Finally in Section 5 we draw the main conclusions.

Our numerical methods were implemented in MATLAB© and FREEFEM++, the experiments were carried out in an Intel(R) Core(TM)2 Duo CPU E6850 @ 3.00 GHz. The codes for the numerical tests are available on request.

## 2. Extinction-time for a Lotka–Volterra predator–prey model

### 2.1. The Lotka–Volterra model

A classical model in mathematical ecology is the standard Lotka–Volterra predator–prey model. Let us consider a simple ecosystem consisting of preys (for example rabbits) that have an infinite supply of food and predators (for example foxes) that need prey for their food. Following [6, p. 359] or [5, p. 149], this problem is modeled by a nonlinear SDE system:

$$\begin{cases} dR = R(2 - \alpha F)dt + \sqrt{R(2 + \alpha F)}dW_1, & t > 0, \\ dF = F(\alpha R - 1)dt + \sqrt{F(\alpha R + 1)}dW_2, & t > 0, \\ R(0) = r, \quad F(0) = f, \end{cases} \quad (2)$$

where  $R = R(t)$  and  $F = F(t)$  are random variables corresponding to the number of preys and predators, respectively,  $r$  and  $f$  are the initial population sizes,  $\alpha$  is a positive constant and  $W_j(t)$ , for  $j = 1, 2$ , are two independent Wiener processes.

If  $\alpha = 0$ , the two populations do not interact and the foxes die off from starvation. If  $\alpha > 0$ , the foxes encounter the rabbits with a probability that is proportional to the product of their numbers. The deterministic part is very simple: its phase plane has a center in  $(1/\alpha, 2/\alpha)$ , and the solutions are always periodic with a period that depends on the initial conditions (see left hand-side of Fig. 1). These solutions cannot be expressed in terms of other known functions and the equations must be solved numerically. Moreover, as was already commented in [14], even tiny amounts of environmental noise cause huge differences between the stochastic model (2) and the associated deterministic model. On the right hand-side of Fig. 1, we have plotted three types of solutions corresponding to the initial values  $R(0) = 100$ ,  $F(0) = 150$ : the deterministic solution (blue), a stochastic path for  $0 \leq t \leq 20$  (black), and the mean of hundred stochastic paths for  $0 \leq t \leq 12$  (green).

The mean extinction-time  $\tau = \tau(r, f)$  for (2) satisfies the following stationary backward Kolmogorov equation:

$$r(2 - \alpha f) \frac{\partial \tau}{\partial r} + \frac{1}{2} r(2 + \alpha f) \frac{\partial^2 \tau}{\partial r^2} + f(\alpha r - 1) \frac{\partial \tau}{\partial f} + \frac{1}{2} f(\alpha r + 1) \frac{\partial^2 \tau}{\partial f^2} = -1. \quad (3)$$

Taking into account that the number of preys and predators cannot exceed some positive values  $M_r$  and  $M_f$ , respectively, we have the following boundary conditions:

$$\begin{cases} \tau(0, f) = \tau(r, 0) = 0, & (r, f) \in (0, M_r) \times (0, M_f), \\ \frac{\partial \tau}{\partial r}(M_r, f) = 0, & f \in (0, M_f), \\ \frac{\partial \tau}{\partial f}(r, M_f) = 0, & r \in (0, M_r). \end{cases} \quad (4)$$

We can extend this procedure to higher-order moments  $\tau^p = \tau^p(r, f)$ . More precisely, it can be easily shown that  $\tau^p$  satisfy the following partial differential equation

$$r(2 - \alpha f) \frac{\partial \tau^p}{\partial r} + \frac{1}{2} r(2 + \alpha f) \frac{\partial^2 \tau^p}{\partial r^2} + f(\alpha r - 1) \frac{\partial \tau^p}{\partial f} + \frac{1}{2} f(\alpha r + 1) \frac{\partial^2 \tau^p}{\partial f^2} = -p\tau^{p-1} \quad (5)$$

with the above boundary conditions (4) written for  $\tau^p$ .

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