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Convergence and asymptotic stability of the explicit Steklov method for stochastic differential equations

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ABSTRACT

In this paper, we develop a new numerical method with asymptotic stability properties for solving stochastic differential equations (SDEs). The foundations for the new solver are the Steklov mean and an exact discretization for the deterministic version of the SDEs. Strong consistency and convergence properties are demonstrated for the proposed method. Moreover, a rigorous linear and nonlinear asymptotic stability analysis is carried out for the multiplicative case in a mean-square sense and for the additive case in a path-wise sense using the pullback limit. In order to emphasize the characteristics of the Steklov discretization we use as benchmarks the stochastic logistic equation and the Langevin equation with a nonlinear potential of the Brownian dynamics. We show that the Steklov method has mild stability requirements and allows long-time simulations in several applications.

1. Introduction

In the last decades, stochastic differential modeling has become a rapidly-growing research area. Historically, it appeared as an extension of the deterministic differential modeling of overidealized situations with fluctuating behavior of the analyzed physical phenomenon. Actually, it is an important research area by itself that describes important phenomena such as turbulent diffusion, spread of diseases, genetic regulation, and motion of particles [1–3]. However, analytic solutions of ordinary stochastic differential equations are more difficult to obtain than in the deterministic case. Thus, the theory of stochastic numerical integration has almost simultaneously been developed. The first developed numerical methods for SDE were stochastic extensions of deterministic algorithms like the schemes of: Euler–Maruyama, Taylor, Runge–Kutta, [4–6]. Nevertheless, new methods have been derived according to the structural or dynamic properties of the SDE. Some examples are the balanced methods for stiff SDE [7] or quasi-symplectic methods for stochastic Hamiltonian systems [8].

Stochastic numerical models may facilitate the analysis of some properties that are difficult or impossible to measure experimentally in laboratories as well as to simulate their asymptotic behavior. In these cases, harsh dynamic stability properties are required such as the asymptotic stability in mean and mean-square. In the last decade, several works related to linear stability for the most common methods for SDE have been published [9–11]. A linear analysis can be considered as a first step for understanding the method, but it is not an indicator of the qualitative behavior of the method in a nonlinear case [12]. Thus, some theoretical work on asymptotic stability has appeared for nonlinear SDE with multiplicative noise by Bokor [4] and with additive noise by Buckwar et al. [13]. Since, most popular schemes are stochastic extensions of the deterministic counterpart, sometimes their asymptotic stability conditions are very restrictive. Consider for example in Brownian dynamics simulations where a classical Euler discretization, *CBD* method, is the standard method to solve the Langevin equation [14–16]. The step size of the time integration for the CBD method has to be pint-size, otherwise the scheme becomes

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unstable. Another example with stability challenges is in the numerical approximation of the logistic SDE, since it requires implicit approximations with some monotone properties [17]. In this work, we focus on the following stochastic differential equation

$$dX_t = F(t, X_t)dt + G(t, X_t)dB_t, \qquad X_0 = x_0 \tag{1}$$

considering the drift term as $F(t, X_t) = F_1(t)F_2(X_t)$. The vast majority of phenomena are modeled with an autonomous deterministic term, so this formulation of F is not too restrictive. Given this functional form of F, we propose an exact explicit algorithm for solving the deterministic equation linked to (1); details of this exact differentiation are given in [18]. So, the main characteristic of this new method is that it preserves qualitative features of the deterministic solution associated to the SDE. Next, we prove strong consistency, convergence and study the linear stability of the proposed method using properties of the *Steklov mean* [19]. Moreover, we analyze the nonlinear stability of the Steklov stochastic approximation specifically the asymptotic mean-square stability in the multiplicative case and the path-wise stability in the additive case. Finally, we show the efficiency of the new scheme in numerical problems with harsh requirements of stability like the logistic equation for the multiplicative case and the Langevin equation with a particular potential for the additive case.

This paper is organized as follows: In Section 2, we construct the explicit Steklov method for the SDE (1) and show its development with some examples. In the next section, we prove strong consistency and convergence of the new explicit method. In Section 4, sufficient conditions for the asymptotic mean and mean-square stability are given for both additive and multiplicative cases. A nonlinear stability analysis is carried out in Section 6, where we prove that the explicit Steklov approximation is asymptotically stable in square mean sense in the multiplicative case and it is path-wise stable under certain conditions in the additive case. In Section 7, we test the Steklov method for the stochastic logistic equation in the multiplicative case and for the Langevin equation in Brownian dynamics. Also, we show numerical evidence that the Steklov method is successful with step sizes significantly large reaching larger time scales of simulation. Finally, we give some conclusions.

2. Steklov method

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ a filtered and complete probability space with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian process B_t. Then the one-dimensional SDE (1) with constant initial condition has a unique solution if F and G are globally Lipschitz-continuous functions satisfying the following linear growth conditions for a positive constant M:

$$|F(x, t)| \le M(1 + |x| + |t|), \qquad |G(x, t)| \le M(1 + |x| + |t|),$$

for all $x \in \mathbb{R}$ and $t \in [0, T]$, see [6]. Under these considerations we construct the Steklov numerical scheme for the SDE (1) based on its integral formulation:

$$X_t = X_0 + \int_0^t F(s, X_s)ds + \int_0^t G(s, X_s)dB_s, \quad t \in [0, T],\ X_0 = x_0, \tag{2}$$

where X_t denotes the value of the process at time t with initial value X_0. First we discretize the time domain with a uniform step size h such that $t_n = nh$ for $n = 0, 1, 2, \dots, N$ and denote by Y_n the numerical solution at t_n. Now we approximate the stochastic integral of (2) with the usual form:

$$\int_{t_n}^{t_{n+1}} G(s, X_s)dB_s \approx G(t_n, Y_n)\Delta B_n, \quad \Delta B_n := (B_{t_{n+1}} - B_{t_n}) = \sqrt{h}V_n, \tag{3}$$

where $B_{t_{n+1}} - B_{t_n}$ is a discrete standard Brownian motion such that $V_n \sim \mathcal{N}(0, 1)$. We can obtain different schemes depending on the numerical integration used for the first integral of (2). For example, if we choose Euler's approximation:

$$\int_{t_n}^{t_{n+1}} F(s, X_s)ds \approx F(t_n, Y_n)(t_{n+1} - t_n), \tag{4}$$

then we obtain the Euler–Maruyama scheme as follows:

$$Y_{n+1} = Y_n + F(t_n, Y_n)h + G(t_n, Y_n)\Delta B_n, \quad n = 1, \dots, N-1, \qquad Y_0 = x_0. \tag{5}$$

Assuming that we can rewrite the function F as $F(t, X_t) = F_1(t)F_2(X_t)$, we propose an alternative approach to (4) based on the construction of an exact discretization for the deterministic differential equation associated to (1):

$$\frac{dx}{dt} = F_1(t)F_2(x), \qquad x(0) = x_0. \tag{6}$$

Integrating this equation in the interval $[t_n, t_{n+1})$ and using the Steklov mean [18], we have

$$\int_{t_n}^{t_{n+1}} F_1(s)F_2(x)ds \approx \phi_1(t_n)\phi_2(y_n, y_{n+1})(t_{n+1} - t_n), \tag{7}$$

where

$$\phi_1(t_n) = \frac{1}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} F_1(s)ds \quad \text{and} \quad \phi_2(y_n, y_{n+1}) = \left(\frac{1}{y_{n+1} - y_n}\int_{y_n}^{y_{n+1}} \frac{du}{F_2(u)}\right)^{-1}.$$