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Constructing positive reliable numerical solution for American call options: A new front-fixing approach

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ABSTRACT

A new front-fixing transformation is applied to the Black–Scholes equation for the American call option pricing problem. The transformed non-linear problem involves homogeneous boundary conditions independent of the free boundary. The numerical solution by an explicit finite-difference method is positive and monotone. Stability and consistency of the scheme are studied. The explicit proposed method is compared with other competitive implicit ones from the points of view accuracy and computational cost.

1. Introduction

Free boundary problems appear in plasma physics, semiconductors, financial markets and other fields [1–3]. The free boundary has to be determined as a part of the solution. Crank in [2] systematized the knowledge about moving and free boundary problems and presented a front-fixing method for such problems. The method is based on Landau's transform [4] that let the unknown boundary be included into equation in exchange for a fixed boundary.

American option pricing leads to the free boundary problem [5]. Wu and Kwok in [6] introduced a logarithmic front-fixing transformation for solving such problems to the field of option pricing. Recently this technique has been treated in [7–9]. Another transformation related to a synthetic portfolio is presented in [10,11] involving the first spatial derivative of the option price. The transformed equation can be numerically solved by a finite element method (see [12]).

In this paper we introduce a new front-fixing transformation for American call option on dividend-paying assets. Under this transformation a nonlinear PDE with homogeneous boundary conditions independent of the free boundary is obtained. This fact simplifies the numerical analysis of the finite difference scheme. The proposed explicit finite difference scheme preserves theoretical properties of the solution mentioned in [13]. Dealing with prices it is important to guarantee that the proposed numerical solutions be non-negative. Our scheme guarantees this property as well as monotonicity of the free boundary and the option price. Numerical experiments show that the method is efficient and accurate in comparison with other implicit methods.

The paper is organized as follows. In Section 2 we introduce a new front-fixing transformation for the American call option problem and an explicit finite difference scheme is constructed. In Section 3 we study properties of the numerical solution, such as the non-negativity and monotonicity of the option price, increasing monotonicity and concave behaviour of the optimal exercise boundary. In Section 4 stability and consistency are treated. In last section we present implicit scheme and compare proposed method with other approaches as well as illustrate efficiency and convergence of the method.

Throughout the paper we will denote for a given $x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{R}^N$ its supremum norm as $\|x\|_\infty = \max\{|x_i| : 1 \le i \le N\}$.

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2. Front-fixing method

In this section we introduce a new front-fixing transformation similar to the ones used by [6,10,5]. This transformation translate the moving domain to the fixed one and changes the boundary conditions on the left boundary to the homogeneous ones. It allows to apply finite-difference method for the numerical solution. The discretization of the transformed problem and constructing the explicit finite-difference method are presented in this section.

American call option price model is given by [5] as the moving free boundary PDE

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC, \quad 0 < S < B(\tau),\ 0 < \tau \le T, \tag{2.1}$$

together with the boundary and initial conditions

$$C(S,0) = \max(S-E,0), \tag{2.2}$$

$$\frac{\partial C}{\partial S}(B(\tau),\tau) = 1, \tag{2.3}$$

$$C(B(\tau),\tau) = B(\tau) - E, \tag{2.4}$$

$$C(0,\tau) = 0, \tag{2.5}$$

$$B(0) = \begin{cases} E, & r \le q, \\ \frac{r}{q}E, & r > q \end{cases} \tag{2.6}$$

where $\tau = T - t$ denotes the time to maturity T, S is the asset's price, $C(S,\tau)$ is the option price, $B(\tau)$ is the unknown early exercise boundary, σ is the volatility of the asset, r is the risk free interest rate, q is the continuous dividend yield and E is the strike price.

It is well known that if there is no dividend payment ($q = 0$), then the optimal strategy is to exercise option at the maturity (see [5, Chapter 7.7], [14]). In that case the American call becomes European one. Because of that we consider problem (2.1)–(2.6) with $q > 0$ [14].

Let us consider the dimensionless transformation with two targets: to fix the computational domain as in [6] and to simplify the boundary conditions like [5, p. 122],

$$x = \ln\frac{B(\tau)}{S}, \qquad c(x,\tau) = \frac{C(S,\tau) - S + E}{E}, \qquad S_f(\tau) = \frac{B(\tau)}{E}. \tag{2.7}$$

Under transformation (2.7) the problem (2.1)–(2.6) can be rewritten in normalized form

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial x^2} - \left(r - q - \frac{\sigma^2}{2} + \frac{S_f'}{S_f}\right)\frac{\partial c}{\partial x} - rc - qS_f e^{-x} + r, \quad x > 0,\ 0 < \tau \le T, \tag{2.8}$$

with new initial and boundary conditions

$$c(x,0) = \begin{cases} 1 - e^{-x}, & r \le q, \\ g(x), & r > q, \end{cases} \quad x \ge 0, \tag{2.9}$$

$$g(x) = \max\left(1 - \frac{r}{q}e^{-x}, 0\right), \tag{2.10}$$

$$\frac{\partial c}{\partial x}(0,\tau) = 0, \tag{2.11}$$

$$c(0,\tau) = 0, \tag{2.12}$$

$$\lim_{x\to\infty} c(x,\tau) = 1, \tag{2.13}$$

$$S_f(0) = \begin{cases} 1, & r \le q, \\ \frac{r}{q}, & r > q. \end{cases} \tag{2.14}$$

Following the ideas of [6,3] and in order to solve the numerical difficulties derived from the discretization at the numerical boundary, we assume that (2.8) holds true at $x = 0$,

$$\frac{\sigma^2}{2}\frac{\partial^2 c}{\partial x^2}(0^+,\tau) - qS_f(\tau) + r = 0. \tag{2.15}$$

Equation (2.8) is a non-linear differential equation on the domain $(0,\infty)\times(0,T]$. In order to solve numerically problem (2.8)–(2.14) at the point (x,τ) in the domain $(0,\infty)\times(0,T]$, one has to consider a bounded numerical domain. Let us introduce $x_{\max}$ large enough to translate the boundary condition (2.13). Then the problem (2.8)–(2.14) can be numerically studied on the fixed domain $[0,x_{\max}]\times[0,\tau]$. The value $x_{\max}$ is chosen following the criterion pointed out in [15].