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Semidefinite inverse eigenvalue problems with prescribed entries and partial eigendata

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ABSTRACT

In this paper, we study the semidefinite inverse eigenvalue problem of reconstructing a real n-by-n matrix C such that it is nearest to the original pre-estimated real n-by-n matrix C_o in the Frobenius norm and satisfies the measured partial eigendata, where the required matrix C should preserve the symmetry, positive semidefiniteness, and the prescribed entries of the pre-estimated matrix C_o. We propose the alternating direction method of multipliers for solving the semidefinite inverse eigenvalue problem, where three related iterative algorithms are presented. We also extend our method to the case of lower bounds. Numerical experiments are reported to illustrate the efficiency of the proposed method for solving semidefinite inverse eigenvalue problems.

1. Introduction

We consider the following semidefinite inverse eigenvalue problem with prescribed entries and partial eigendata:

PESDIEP. Given a pre-estimated real symmetric and positive definite matrix $C_o \in \mathbb{R}^{n\times n}$ *and a set of measured eigendata* $\{(\lambda_k, \mathbf{x}_k) \in \mathbb{R} \times \mathbb{R}^n\}_{k=1}^{p} (p \ll n)$, *find a matrix* $C \in \mathbb{R}^{n\times n}$ *such that it is closest to the original matrix* C_o *in the Frobenius norm, satisfies the measured eigendata* $\{(\lambda_k, \mathbf{x}_k)\}_{k=1}^{p}$, *and retains the symmetry, positive semidefiniteness and prescribed entries* $\{(C_o)_{ij} \mid (i,j) \in \mathcal{I}_s\}$ *of the pre-estimated matrix* C_o, *where* $\mathcal{I}_s \subset \mathcal{N} := \{(i,j) \mid i,j = 1,\ldots,n\}$ *is a given index subset such that* $(j,i) \in \mathcal{I}_s$ if $(i,j) \in \mathcal{I}_s$.

The PESDIEP is a kind of structured inverse eigenvalue problems, which arise in many applications such as structural dynamics, vibrations, control design, circuit theory, inverse Sturm–Liouville problems, applied physics, finite element model updating, etc. For the applications, mathematical theory, and numerical methods on structured inverse eigenvalue problems, one may refer to, for instance, the survey papers [1,2] and the books [3–6] and references therein.

The PESDIEP plays an important role in many applications such as the finite element model updating in structural dynamics and vibration [3–5,7,8]. In practice, the pre-estimated analytic matrix C_o is a physical matrix whose entries involves some physical parameters such as mass, stiffness, length, elasticity, inductance, capacitance, etc. In general, the physical analytical matrix C_o possesses some specific structural constraints (e.g., symmetry, definiteness, sparsity or bandedness). However, the natural frequencies and mode shapes (i.e., eigenvalues of eigenvectors) predicted by the analytic matrix C_o

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often do not match with experimentally measured frequencies and mode shapes. To ensure the validity of the original model, one may update or correct the original analytic matrix C_o via the prescribed partial eigendata, which can be experimentally measured from a practical structure. It is desired to update the original matrix C_o with minimal changes. This requires that the updated matrix C should be closest to the original analytic matrix C_o, say, in the Frobenius norm and satisfies the measured eigendata. More importantly, the updated matrix C should preserve various structural constraints of C_o simultaneously. That is, the corrected matrix C should keep the symmetry, definiteness, and sparsity (i.e., prescribed entries) of the analytic matrices C_o unchanged.

Let

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^{p\times p}, \qquad X = [\mathbf{x}_1, \ldots, \mathbf{x}_p] \in \mathbb{R}^{n\times p}.$$

Then, the PESDIEP is to solve the following minimization problem.

$$\begin{array}{ll} \min & \frac{1}{2}\|C - C_o\|^2 \\ \text{subject to (s.t.)} & CX = X\Lambda, \\ & C_{ij} = (C_o)_{ij} \quad \forall (i,j) \in \mathcal{I}_s, \\ & C \in \mathcal{S}^n_+, \end{array} \tag{1}$$

where $\|\cdot\|$ denotes the Frobenius matrix norm or the Euclidean vector norm and $\mathcal{S}^n$ and $\mathcal{S}^n_+$ denote the set of all $n \times n$ real symmetric matrices and the set of all $n\times n$ real symmetric and positive semidefinite matrices, respectively. Without causing any confusion, we regard the minimization problem (1) as the PESDIEP.

As noted in [9], one may find a solution to the PESDIEP (1) by using classical semidefinite programming (SDP) techniques (see for instance [10–12]). However, the primal–dual interior-point methods may not be effective for solving large-scale semidefinite programming problems [12]. In many applications, the problem size of the PESDIEP is very large (say, $n \geq 1000$). In this case, the number np of linear constraints in the PESDIEP (1) is much large even when the number p of given eigenpairs is small (e.g., when $n \geq 1000$ and $p = 30$, $np \geq 30,000$). By dropping the requirement of partial entries (i.e., $C_{ij} = (C_o)_{ij}\ \forall (i,j) \in \mathcal{I}_s$), one may solve the simplified version of PESDIEP (1) by the semismooth Newton method proposed in [13,14]. But the requirement of prescribed entries is vital for practical applications, e.g., it is essential to preserve the sparsity of the original physical matrix C_o. In this case, the number $|\mathcal{I}_s|$ of prescribed entries is very large.

Recently, the alternating direction method of multipliers (ADMM), which is proposed by Glowinski and Marrocco [15], has been used in many areas: optimization, image processing and statistical learning, etc. One may refer to the survey paper [16] and references therein for the applications of the ADMM. In this paper, we propose several iterative algorithms based on the ADMM for solving the PESDIEP. This is motivated by the recent papers due to He, Xu, and Yuan [17] and Zhao, Bai, and Chen [18]. In [17], He, Xu, and Yuan introduced an ADMM for solving large-scale semidefinite programming. In [18], the ADMM is successfully applied to nonnegative inverse eigenvalue problems with partial eigendata. We shall present three ADMM-based iterative algorithms for solving the PESDIEP by adding two auxiliary matrix variables so that the resulted two subproblems can be handled easily, where one of the subproblems has closed-form solution and the other is a quadratic minimization problem which can be solved efficiently by solving its dual problem. We also extend the proposed method to the case of lower bounds. We report some numerical tests, including the comparison with the interior-point approach mentioned in [9,12] for solving the PESDIEP, to illustrate the effectiveness of our method.

Throughout the paper, we use the following notations. The symbol A^T denotes the transpose of a matrix A. I is the identity matrix of an appropriate dimension. Let $\|\cdot\|_{\max}$ denote the entry of largest absolute value of a matrix. Let $\mathcal{D} \subseteq \mathbb{R}^{n\times n}$ (or $\mathcal{S}^n$) be a closed convex set and $\Pi_{\mathcal{D}}\{\cdot\}$ denote the metric projection onto $\mathcal{D}$.

The remaining part of the paper is organized as follows. In Section 2 we propose several ADMM-based iterative algorithms for solving the PESDIEP. In Section 3 we discuss some extensions. In Section 4 we report some numerical tests.

2. An alternating direction method of multipliers

2.1. Problem reformulation

Let $\mathcal{S}^n$ be equipped with the Frobenius inner product

$$\langle A, B\rangle = \mathrm{tr}(AB) \quad \forall A, B \in \mathcal{S}^n,$$

where "tr" means the trace of a matrix. To apply the ADMM to the PESDIEP, one possible way is to rewrite the PESDIEP (1) in the following form:

$$\begin{array}{ll} \min & \frac{1}{2}\|C - C_o\|^2 + \frac{1}{2}\|Y - C_o\|^2 \\ \text{s.t.} & C - Y = 0, \\ & C \in \mathcal{S}^n_+, \quad Y \in S_B, \end{array} \tag{2}$$

where $S_B := \{Y \in \mathcal{S}^n \mid YX = X\Lambda,\ Y_{ij} = (C_o)_{ij}\ \forall (i,j) \in \mathcal{I}_s\}$.