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Convergence rate of weak Local Linearization schemes for stochastic differential equations with additive noise

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ABSTRACT

There exists a diversity of weak Local Linearization (LL) schemes for the integration of stochastic differential equations with additive noise, which differ in the algorithms employed for the numerical implementation of the weak Local Linear discretizations. Despite convergence results for these discretizations have been already developed, the convergence of the weak LL schemes has not been considered up to date. In this work, a general result concerning the convergence rate of the weak LL schemes is presented, as well as specificities for a number of particular schemes. As an application, the convergence of weak LL schemes for equations driven by Poisson processes is presented in addition.

1. Introduction

The evaluation of Wiener functional space integrals and the estimation of diffusion processes are essential matters for the resolution of a number of problems in mathematical physics, biology, finance and other fields. In the solution of this kind of problems, weak numerical integrators for Stochastic Differential Equations (SDEs) have become an important tool [1–5]. Well-known are, for instance, the Euler, the Milstein, the Talay–Tubaro extrapolation, the Runge–Kutta and the Local Linearization methods (see [6] for a review of these methods).

Specifically, weak Local Linearization (LL) schemes for SDEs with additive noise have played a prominent role in the construction of effective inference methods for SDEs [7–11], in the estimation of distribution functions in Monte Carlo Markov Chain methods [12–14] and the simulation of likelihood functions [15]. Extensive simulation studies carried out in these papers have showed that these LL schemes possess high numerical stability and remarkable computational efficiency. Other distinctive feature of the weak LL integrators is that they preserve the ergodicity and geometric ergodicity properties of a wide class of nonlinear SDEs [14].

This paper deals with an open problem related with the weak Local Linearization method. It has been proved that the order-β weak Local Linear discretization is the base for the construction and study of such a method [16]. Based on this discretization, a variety of numerical schemes can be derived, which mainly differ in the algorithm employed for the numerical implementation of the discretization. This feature provides flexibility to the LL method for suitable adjustments when it is applied to certain types of equations (e.g., large systems of SDEs, etc.). However, in contrast with the weak Local Linear discretization, the convergence rate of LL schemes has not been considered so far. This essential issue must be addressed for developing computationally efficient weak LL schemes.

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In this work, a main theorem about the convergence rate of weak LL schemes for SDEs with additive noise is derived. Based on this general result, convergence results for some specific LL schemes like those based on Padé and Krylov–Padé approximations are also obtained. As straightforward application, the convergence rate of some weak LL schemes for stochastic differential equations with jumps is also presented. Numerical simulations are also given in order to illustrate the proven theoretical results and the feasibility of the numerical schemes. A summary of basic results about the LL method is presented for supporting the subsequent presentation.

2. Notations and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\{\mathcal{F}_t,\ t \geq t_0\}$ be an increasing right continuous family of complete sub σ-algebras of $\mathcal{F}$. Consider a d-dimensional diffusion process $\mathbf{x}$ defined by the following stochastic differential equation with additive noise

$$d\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x}(t))dt + \sum_{i=1}^{m} \mathbf{g}_i(t)d\mathbf{w}^i(t) \tag{1}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where the drift coefficient $\mathbf{f} : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion coefficient $\mathbf{g}_j : [t_0, T] \to \mathbb{R}^d$ are differentiable functions, $\mathbf{w} = (\mathbf{w}^1, \ldots, \mathbf{w}^m)$ is an m-dimensional $\mathcal{F}_t$-adapted standard Wiener process, and $\mathbf{x}_0$ is a $\mathcal{F}_{t_0}$-measurable random vector. The standard conditions for the existence and uniqueness of a solution for (1)–(2) are assumed.

Consider the time discretization $(t)_h = \{t_n : n = 0, 1, \ldots, N\}$, with maximum step-size $h \in (0, 1)$, defined as a sequence of $\mathcal{F}$-stopping times that satisfy $t_0 < t_1 < \cdots < t_N = T$ and $\sup_n(h_n) \leq h$, w.p.1, where t_n is $\mathcal{F}_{t_n}$-measurable for each $n = 0, 1, \ldots, N$, and $h_n = t_{n+1} - t_n$. In addition, let us denote $n_t = \max\{n = 0, 1, 2, \ldots :\ t_n \leq t \text{ and } t_n \in (t)_h\}$ for all $t \in [t_0, T]$.

2.1. Weak Local Linear discretization

The following basic results about the Local Linearization method are taken from [16].

Definition 1. For a given time discretization $(t)_h$, the order-β $(=1, 2)$ weak Local Linear discretization of the solution of (1)–(2) is defined by the recurrent relation

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \phi_\beta(t_n, \mathbf{y}_n; h_n) + \eta(t_n, \mathbf{y}_n; h_n), \tag{3}$$

where

$$\phi_\beta(t_n, \mathbf{y}_n; h_n) = \int_0^{h_n} e^{\mathbf{f_x}(t_n, \mathbf{y}_n)(h_n - s)}(\mathbf{f_x}(t_n, \mathbf{y}_n)\mathbf{y}_n + \mathbf{a}_n^\beta(t_n + s))ds, \tag{4}$$

and $\eta\,(t_n, \mathbf{y}_n; h)$ is a zero mean Gaussian random variable with variance

$$\mathbf{\Sigma}(t, \mathbf{y}; \delta) = \int_0^{\delta} e^{\mathbf{f}_x(t, \mathbf{y})(\delta - s)}\mathbf{G}(t+s)\mathbf{G}^\intercal(t+s)e^{\mathbf{f}_\mathbf{x}^\intercal(t, \mathbf{y})(\delta - s)}ds. \tag{5}$$

Here, $\mathbf{G}(u) = [\mathbf{g}_1(u), \ldots, \mathbf{g}_m(u)]$ is a $d \times m$ matrix,

$$\mathbf{a}_n^\beta\,(u) = \begin{cases} \mathbf{f}(t_n, \mathbf{y}_n) - \mathbf{f_x}(t_n, \mathbf{y}_n)\mathbf{y}_n + \mathbf{f}_t(t_n, \mathbf{y}_n)(u - t_n) & \text{for } \beta = 1 \\ \mathbf{a}_n^1\,(u) + \dfrac{1}{2}\displaystyle\sum_{j=1}^{m}(\mathbf{I}_d \otimes \mathbf{g}_j^\intercal\,(t_n))\mathbf{f_{xx}}(t_n, \mathbf{y}_n)\mathbf{g}_j\,(t_n)\,(u - t_n) & \text{for } \beta = 2, \end{cases}$$

$\mathbf{f_x}$, $\mathbf{f}_t$ denote the partial derivatives of $\mathbf{f}$ with respect to the variables $\mathbf{x}$ and t, respectively, $\mathbf{f_{xx}}$ the Hessian matrix of $\mathbf{f}$ with respect to $\mathbf{x}$, $\mathbf{I}_d$ is the d-dimensional identity matrix, and the initial point $\mathbf{y}_0$ is assumed to be a $\mathcal{F}_{t_0}$-measurable random vector.

Denote by $\mathcal{C}_P^l$ the space of l time continuously differentiable functions with partial derivatives up to order l having polynomial growth.

Theorem 2. *Let* $\mathbf{x}$ *be the solution of the SDE* (1)–(2), *and* $\mathbf{y}$ *the order-*β *weak Local Linear discretization of* $\mathbf{x}$ *defined by* (3). *Suppose that the drift and diffusion coefficients of the SDE* (1) *satisfy the following conditions*

$$\mathbf{f}^k \in \mathcal{C}_P^{2(\beta+1)}([t_0, T] \times \mathbb{R}^d, \mathbb{R}) \quad \text{and} \quad \mathbf{g}_i^k \in \mathcal{C}_P^{2(\beta+1)}([t_0, T], \mathbb{R}) \tag{6}$$

$$|\mathbf{f}(s, \mathbf{u})| + \sum_{i=1}^{m} |\mathbf{g}_i(s)| \leq K\,(1 + |\mathbf{u}|)\,, \tag{7}$$