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Effective transmission conditions for Hamilton–Jacobi equations defined on two domains separated by an oscillatory interface

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A B S T R A C T

We consider a family of optimal control problems in the plane with dynamics and running costs possibly discontinuous across an oscillatory interface Γ_ε. The oscillations of the interface have small period and amplitude, both of the order of ε, and the interfaces Γ_ε tend to a straight line Γ. We study the asymptotic behavior as $\varepsilon \to 0$. We prove that the value function tends to the solution of Hamilton–Jacobi equations in the two half-planes limited by Γ, with an effective transmission condition on Γ keeping track of the oscillations of Γ_ε.

R É S U M É

On considère une famille de problèmes de contrôle optimal dans le plan, avec des dynamiques et des coûts instantannés discontinus à la traversée d'une interface oscillante Γ_ε. Les oscillations de l'interface sont périodiques, et leur amplitude et période sont de l'ordre de ε, où ε est un petit paramètre. Quand ε tend vers 0, Γ_ε tend vers une droite Γ. On étudie le comportement asymptotique de la fonction valeur du problème. On démontre que cette dernière tend vers la solution d'une équation de Hamilton–Jacobi posée dans les deux demi-plans séparés par Γ, avec une condition de transmission effective sur Γ dont le hamiltonien garde une trace des oscillations évanescentes.

1. Introduction

The goal of this paper is to study the asymptotic behavior as $\varepsilon \to 0$ of the value function of an optimal control problem in $\mathbb{R}^2$ in which the running cost and dynamics may jump across a periodic oscillatory interface Γ_ε, when the oscillations of Γ_ε have a small amplitude and period, both of the order of ε. The interface Γ_ε separates two unbounded regions of $\mathbb{R}^2$, Ω_ε^L and Ω_ε^R. To characterize the optimal control

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problem, one has to specify the admissible dynamics at a point $x \in \Gamma_\varepsilon$: in our setting, no mixture is allowed at the interface, i.e. the admissible dynamics are the ones corresponding to the subdomain Ω_ε^L **and** entering Ω_ε^L, or corresponding to the subdomain Ω_ε^R **and** entering Ω_ε^R. Hence the situation differs from those studied in the articles of G. Barles, A. Briani and E. Chasseigne [5,6] and of G. Barles, A. Briani, E. Chasseigne and N. Tchou [7], in which mixing is allowed at the interface. The optimal control problem under consideration has been first studied in [16]: the value function is characterized as the viscosity solution of a Hamilton–Jacobi equation with special transmission conditions on Γ_ε; a comparison principle for this problem is proved in [16] with arguments from the theory of optimal control similar to those introduced in [5,6]. In parallel to [16], Imbert and Monneau have studied similar problems from the viewpoint of PDEs, see [11], and have obtained comparison results for quasi-convex Hamiltonians. In particular, [11] contains a characterization of the viscosity solution of the transmission problem with a reduced set of test-functions; this characterization will be used in the present work. Note that [16,11] can be seen as extensions of articles devoted to the analysis of Hamilton–Jacobi equations on networks, see [1,13,2,12], because the notion of interface used there can be seen as a generalization of the notion of vertex (or junction) for a network.

We will see that as ε tends to 0, the value function converges to the solution of an effective problem related to a flat interface Γ, with Hamilton–Jacobi equations in the half-planes limited by Γ and a transmission condition on Γ.

Whereas the partial differential equation far from the interface is unchanged, the main difficulty consists in finding the effective transmission condition on Γ. Naturally, the latter depends on the dynamics and running costs but also keeps memory of the vanishing oscillations. The present work is closely related to two recent articles, [3] and [10], about singularly perturbed problems leading to effective Hamilton–Jacobi equations on networks. Let us briefly describe the singular perturbation problems studied in [3] and [10]: in [3], some of the authors of the present paper study a family of star-shaped planar domains D^ε made of N non-intersecting semi-infinite strips of thickness ε and of a central region whose diameter is proportional to ε. As $\varepsilon \to 0$, the domains D^ε tend to a network $\mathcal{G}$ made of N half-lines sharing an endpoint O, named the vertex or junction point. For infinite horizon optimal control problems in which the state is constrained to remain in the closure of D^ε, the value function tends to the solution of a Hamilton–Jacobi equation on $\mathcal{G}$, with an effective transmission condition at O. The related effective Hamiltonian, which corresponds to trajectories staying close to the junction point, was obtained in [3] as the limit of a sequence of ergodic constants corresponding to larger and larger bounded subdomains. The same kind of construction was then used in [10], in which Galise, Imbert and Monneau study a family of Hamilton–Jacobi equations in a simple network composed of two half-lines with a perturbation of the Hamiltonian localized in a small region close to the junction.

In the proof of convergence, we will see that the main technical point lies in the construction of correctors and in their use in the perturbed test-function method of Evans, see [8]. As in [3] and [10], an important difficulty comes from the unboundedness of the domain in which the correctors are defined. The strategies for passing to the limit in [3] and [10] differ: the method proposed in [3] consists of constructing an infinite family of correctors related to the vertex, while in [10], only one corrector related to the vertex is needed thanks to the use of the above mentioned reduced set of test-functions. Arguably, the strategy proposed in [3] is more natural and that in [10] is simpler. For this reason, the technique implemented in the present work for proving the convergence to the effective problem will be closer to the one proposed in [10]. Note that similar techniques are used in the very recent work [9], which deals with applications to traffic flows. The question of the correctors in unbounded domains has recently been addressed by P-L. Lions in his lectures at Collège de France, [14], precisely in January and February 2014: the lectures dealt with recent and still unpublished results obtained in collaboration with T. Souganidis on the asymptotic behavior of solutions of Hamilton–Jacobi equations in a periodic setting with some localized defects. Finally, we stress the fact that the technique proposed in the present work is not specific to the transmission condition imposed on Γ_ε.