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An effective Hamiltonian for the eigenvalue asymptotics of the Robin Laplacian with a large parameter

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A B S T R A C T

We consider the Laplacian on a class of smooth domains $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, with attractive Robin boundary conditions:

$$Q_\alpha^\Omega u = -\Delta u, \quad \frac{\partial u}{\partial n} = \alpha u \text{ on } \partial\Omega, \ \alpha > 0,$$

where n is the outer unit normal, and study the asymptotics of its eigenvalues $E_j(Q_\alpha^\Omega)$ as well as some other spectral properties for α tending to $+\infty$. We work with both compact domains and non-compact ones with a suitable behavior at infinity. For domains with compact C^2 boundaries we show that, for each fixed j,

$$E_j(Q_\alpha^\Omega) = -\alpha^2 + \mu_j(\alpha) + \mathcal{O}(\log \alpha),$$

where $\mu_j(\alpha)$ is the jth eigenvalue of the operator $-\Delta_S - (\nu - 1)\alpha H$ with $(-\Delta_S)$ and H being respectively the positive Laplace–Beltrami operator and the mean curvature on $\partial\Omega$. Analogous results are obtained for a class of domains with non-compact boundaries. In particular, we discuss the existence of eigenvalues for non-compact domains and the existence of spectral gaps for periodic domains. We also show that the remainder estimate can be improved under stronger regularity assumptions.

The effective Hamiltonian $-\Delta_S - (\nu - 1)\alpha H$ enters the framework of semi-classical Schrödinger operators on manifolds, and we provide the asymptotics of its eigenvalues for large α under various geometric assumptions. In particular, we describe several cases for which our asymptotics provides gaps between the eigenvalues of Q_α^Ω for large α.

R É S U M É

On considère le laplacien sur une classe de domaines réguliers $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, avec conditions de Robin attractives :

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$$Q_\alpha^\Omega u = -\Delta u, \quad \frac{\partial u}{\partial n} = \alpha u \text{ sur } \partial\Omega,\ \alpha > 0,$$

où n est la normale unitaire sortante, et on étudie le comportement de ses valeurs propres $E_j(Q_\alpha^\Omega)$ ainsi que d'autres propriétés spectrales lorsque α tend vers $+\infty$. On considère des domaines soit compacts, soit non-compacts avec un comportement convenable à l'infini. Pour les domaines C^2 dont le bord est compact, on démontre que pour tout j fixé on a

$$E_j(Q_\alpha^\Omega) = -\alpha^2 + \mu_j(\alpha) + \mathcal{O}(\log \alpha),$$

où $\mu_j(\alpha)$ est la $j$ème valeur propre de l'opérateur $-\Delta_S - (\nu-1)\alpha H$, où $(-\Delta_S)$ et H sont respectivement l'opérateur de Laplace–Beltrami positif et la courbure moyenne sur $\partial\Omega$. Des résultats analogues sont obtenus pour une classe de domaines à bord non-compact. En particulier, on étudie l'existence de valeurs propres pour une classe de domaines noncompacts ainsi que l'existence de trous spectraux pour des domaines périodiques. On montre également que l'estimée du reste peut être améliorée sous des hypothèses de régularité plus fortes.
Le hamiltonien effectif $-\Delta_S - (\nu-1)\alpha H$ entre dans le cadre des opérateurs de Schrödinger semi-classiques sur des variétés, et on décrit l'asymptotique de ses valeurs propres lorsque $\alpha \to +\infty$ pour diverses hypothèses géométriques. En particulier, on dérive des cas pour lequels notre asymptotique donne l'existence de trous entre les valeurs propres de Q_α^Ω pour α suffisamment grand.

1. Introduction

Let $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, be an open set with a sufficiently regular boundary $S := \partial\Omega$. For $\alpha \in \mathbb{R}$, denote by Q_α^Ω the operator $Q_\alpha^\Omega u = -\Delta u$ acting on the functions u defined in Ω and satisfying the Robin boundary condition

$$\frac{\partial u}{\partial n} = \alpha u \text{ on } S,$$

where n is the outer unit normal at S. More precisely, Q_α^Ω is the self-adjoint operator in $L^2(\Omega)$ associated with the quadratic form q_α^Ω defined on the domain $\mathcal{D}(q_\alpha^\Omega) = H^1(\Omega)$ by

$$q_\alpha^\Omega(u,u) = \int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_S u^2 \mathrm{d}S,$$

where $\mathrm{d}S$ stands for the $(\nu-1)$-dimensional Hausdorff measure on S, which is closed and semibounded from below under suitable assumptions (e.g. if S is compact or with a suitable behavior at infinity, see below), and we denote by $E_j(Q_\alpha^\Omega)$ the jth eigenvalue of Q_α^Ω below the bottom of the essential spectrum, as soon as it exists. The aim of the paper is to obtain new results on the asymptotics of the eigenvalues as α tends to $+\infty$.

The problem appears in various applications, such as reaction–diffusion processes [30] and the enhanced surface superconductivity [16], and some results were obtained in previous works. Let us present briefly the state of art for compact domains. It was shown in [30,31] that for piecewise smooth Lipschitz domains Ω one has $E_1(Q_\alpha^\Omega) = -C_\Omega\alpha^2 + o(\alpha^2)$ as $\alpha \to +\infty$, where $C_\Omega \geq 1$ is a constant depending on the geometric properties of Ω. In particular, $C_\Omega = 1$ for C^1 domains, see [6,32]. More detailed asymptotic expansions for some specific non-smooth domains were considered in [22,31,36]. As for smooth domains, a more detailed result was obtained first in [10,35] for $\nu = 2$ and then in [37] for any $\nu \geq 2$: if the domain is C^3 and $j \in \mathbb{N}$ is fixed, then

$$E_j(Q_\alpha^\Omega) = -\alpha^2 - (\nu-1)H_{\max}\alpha + \mathcal{O}(\alpha^{2/3}), \tag{1.1}$$