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Fractal regularity results on optimal irrigation patterns

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ABSTRACT

In this paper the problem of the regularity, i.e. *fractal behaviour*, of the minima of the branched transport problem is addressed. We show that, under suitable conditions on the irrigated measure, the minima present a fractal regularity, that is on a given branch of length l the number of branches bifurcating from it whose length is comparable with ε can be estimated both from above and below by l/ε.

RÉSUMÉ

Dans cet article, le problème de la régularité, c'est-à-dire du *comportement fractal*, des minima du problème de transport branché est considéré. On montre que, dans des conditions appropriées sur la mesure irriguée, les minima présentent une régularité fractale, à savoir sur une branche de longueur l le nombre de branches de bifurcation de celle-ci dont la longueur est comparable à ε peut être estimé à la fois supérieurement et inférieurement en fonction de l/ε.

1. Introduction

1.1. Optimal transport problems

Optimal transport in the Monge–Kantorovich viewpoint. Optimal transport problems were first considered by Monge in 1781. In optimal transport problems the datum is a couple (μ^+, μ^-) of probability measures (respectively named *initial* and *final* measure). The problem is then to minimize

$$M(t) := \int_{\mathbf{R}^N} c\big(x, t(x)\big)\, \mathrm{d}\mu^+(x)$$

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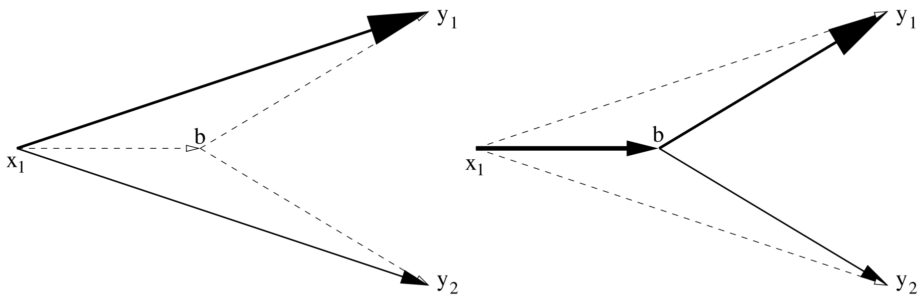

Fig. 1. V-shaped versus Y-shaped transport.

among *transport maps*, i.e. maps $t : \mathbf{R}^N \to \mathbf{R}^N$ such that, for every Borel set B, $\mu^-(B) = \mu^+(t^{-1}(B))$. The function $c : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ is a positive lower semi-continuous function, usually the p-th power of the Euclidean distance.

In 1948 Kantorovich proposed a relaxed version of Monge's problem. Instead of transport maps, *transport plans* are considered, i.e. probability measures $\pi \in \mathcal{P}(\mathbf{R}^N \times \mathbf{R}^N)$ such that $\pi(A \times \mathbf{R}^N) = \mu^+(A)$, $\pi(\mathbf{R}^N \times B) = \mu^-(B)$. The problem is then to minimize

$$K(\pi) = \int\limits_{\mathbf{R}^N \times \mathbf{R}^N} c(x, y)\, \mathrm{d}\pi(x, y).$$

Note that, if t is a transport map, the transport plan defined as

$$\pi_t(C) := \mu^+\big(\big\{x \in \mathbf{R}^N\colon\ \big(x, t(x)\big) \in C\big\}\big)$$

satisfies $K(\pi_t) = M(t)$. Because of this, Kantorovich's Problem extends Monge's one (it is actually its relaxation w.r.t. the weak convergence of measures, see [1]).

Branched transport problems. Branched transport problems were introduced in order to model many artificial and natural systems (like roads, pipelines, bronchial, and cardiovascular ones) which can naturally be viewed as transport problem, but the Monge–Kantorovich setting is not suitable to describe them, since the minimum value of either M or K depends only on the initial and final measure and not on the route the mass covers, so that M or K do not select branched structures.

For example, if we consider the transport problem of a Dirac mass onto the convex combination of two Dirac masses, the solution of the Monge–Kantorovich will be the one on the left of Fig. 1: the initial mass is split and brought on the support of the final measure on a straight line. On the other hand, one would like a functional whose minima were those on the right of Fig. 1, where the mass is not split from the beginning, since in branched transport it is cheaper to move it together as much as possible.

In order to describe such systems, Maddalena, Morel, and Solimini in [12] proposed a model based on the Lagrangian formulation of the fluid flow in a system of pipes. In their approach they consider an appropriate functional defined on curves with a common initial point $S \in \mathbf{R}^N$. Such curves represent the trajectory of fluid particles or veins in the cardiovascular system. In this model the initial measure is then a Dirac mass in S, while the final one is obtained counting how many curves stop in a given volume.

The attempts to model such situations are several (we recommend the book [4] to the interested reader, which offers an exhaustive introduction to this research topic). Let us recall some of them to the reader. The first one are Xia's *transport paths* (see [20]). This model considers a functional which is the relaxation of an appropriate cost defined on weighted directed graphs. Bernot, Caselles, and Morel's *traffic plans* (see [3]) are instead another Lagrangian approach to the problem. In [7] Brancolini, Buttazzo, and Santambrogio introduce *path functionals*, functionals defined on curves in Wasserstein spaces such that curves which are