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Error analysis of first-order projection method for time-dependent magnetohydrodynamics equations

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ABSTRACT

This paper focuses on a linearized fully discrete projection scheme for time-dependent magnetohydrodynamics equations in three-dimensional bounded domain. It is shown that the proposed projection scheme allows for a discrete energy inequality and is unconditionally stable. In addition, we present a rigorous analysis for the rates of convergence.

1. Introduction

The incompressible magnetohydrodynamics (MHD) equations are used to describe the flow of a viscous, incompressible and electrically conducting fluid. For the understanding of the physical background of the MHD equations, we refer to Hughes [10] and Moreau [11]. Let $\Omega \subset \mathbf{R}^3$ be a bounded and simply-connected domain which is either convex or has a $C^{1,1}$ boundary $\partial\Omega$. We consider the non-dimensional MHD equations in the primitive variable formulation

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\mathbf{b}\times \operatorname{curl}\mathbf{b} = \mathbf{f}, \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\mathbf{b} = 0, \tag{1.2}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \frac{1}{Rm}\operatorname{curl}(\operatorname{curl}\mathbf{b}) - \operatorname{curl}(\mathbf{u}\times\mathbf{b}) = 0, \tag{1.3}$$

for $x \in \Omega$ and $t \in (0, T)$ with $T > 0$, where Re, Rm and S are three positive constants and denote the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. The vector-value function $\mathbf{f}$ represents the body forces applied to the fluid. The MHD equations (1.1)–(1.3) couple the incompressible Navier–Stokes equations with Maxwell's equations. Thus, the unknowns in (1.1)–(1.3) are the fluid velocity $\mathbf{u}$, the pressure p and the magnetic field $\mathbf{b}$. The MHD equations should be completed by the appropriate initial and boundary conditions. For the sake of simplicity, we consider the following initial and boundary conditions:

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$$\mathbf{u}(x,0)=\mathbf{u}_0, \quad \mathbf{b}(x,0)=\mathbf{b}_0 \quad \text{in } \Omega, \tag{1.4}$$

$$\mathbf{u}=0, \quad \mathbf{b}\cdot\mathbf{n}=0, \quad \text{curl}\,\mathbf{b}\times\mathbf{n}=0 \quad \text{on } \partial\Omega\times[0,T], \tag{1.5}$$

where $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. The initial vector functions $\mathbf{u}_0$ and $\mathbf{b}_0$ satisfy the compatibility condition div $\mathbf{u}_0=0$ and div $\mathbf{b}_0=0$.

It was observed that testing (1.1) and (1.3) by $\mathbf{u}$ and $S\mathbf{b}$, respectively, and adding the resulting equations leads to the following energy identity:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\mathbf{u}|^2+S|\mathbf{b}|^2\right)dx+\int_\Omega\left(\frac{1}{Re}|\nabla\mathbf{u}|^2+\frac{S}{Rm}|\text{curl}\,\mathbf{b}|^2\right)dx=\int_\Omega \mathbf{f}\cdot\mathbf{u}dx \quad \forall t>0.$$

Thus, for any prescribed initial data $(\mathbf{u}_0,\mathbf{b}_0)\in\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$, the MHD problem (1.1)–(1.5) exists the global weak solutions. On the regularities of the weak solutions, Sermange–Temam in [14] established the existences of local strong solution with large initial data and global strong solution with small initial data.

For the numerical analysis of the MHD problem, the mixed finite element approximations were first proposed and studied for the stationary MHD problem in [7], where $\mathbf{H}^1$-conforming elements were used to discretize the magnetic field provided that Ω is either convex or has a $C^{1,1}$ boundary $\partial\Omega$. Inspired by the absolutely stable methods for Stokes problem in [3], a stabilized mixed finite element method for stationary MHD problem was developed by Gerbeau [4]. For the time-dependent MHD equations (1.1)–(1.3), recently, He proposed a linearized semi-implicit Euler scheme in [8], where $\mathbf{L}^2$-unconditional convergence of this scheme was proved by using the negative norm technique. Other fully discrete Crank–Nicolson schemes were studied in [19,20]. For the non-convex domain or Lipschitz polyhedra domain of engineering practice, the magnetic field $\mathbf{b}$ may have regularity below $\mathbf{H}^1(\Omega)$. In this case, the $\mathbf{H}^1$-conforming finite element discretization for $\mathbf{b}$, albeit stable, may not converge to corresponding magnetic field. A mixed finite element formulation based on $\mathbf{H}$(curl)-elements (or Nédélec elements) for $\mathbf{b}$ was proposed and studied by Schötzau in [13] for the stationary MHD problem. For a review of various numerical methods for the MHD equations, we refer to Gerbeau–Bris–Lelièvre [5].

In the present work, we will consider the three-dimensional time-dependent MHD equations (1.1)–(1.3) with the initial and boundary conditions (1.4) and (1.5), and propose a linearized fully discrete scheme based on a modified Chorin's projection scheme for Navier–Stokes equations in [15]. The projection methods were first proposed by Chorin [2] and Temam [17], and have been further developed in various directions. The main advantage of the projection methods is first to compute a velocity field without taking into account incompressibility, and then perform a pressure correction, which is a projection back to the subspace of solenoidal (divergence-free) vector fields. For a review of projection methods for the Navier–Stokes equations, we refer to [6]. To state our main results derived in this paper, we recall the following terminology [15]:

Definition 1.1. Let X be a Banach space equipped with norm $||\cdot||_X$ and $f:[0,T]\longrightarrow X$ is continuous. Let $0=t_0<t_1<\cdots<t_N=T$ be a uniform partition of the time interval $[0,T]$ with time step $\Delta t=T/N$ and $t_n=n\Delta t$ for $0\le n\le N$. We say $f_{\Delta t}$ is a weakly order α approximation of f in X if there exists a constant C independent of Δt such that

$$\Delta t\sum_{n=0}^{N}||f_{\Delta t}(t_n)-f(t_n)||_X^2\le C(\Delta t)^{2\alpha};$$

and we say $f_{\Delta t}$ is a strongly order α approximation of f in X if there exists a constant C independent of Δt and n such that

$$\max_{0\le n\le N}||f_{\Delta t}(t_n)-f(t_n)||_X^2\le C(\Delta t)^{2\alpha}.$$

To our best knowledge, Prohl in [12] first proposed a projection scheme for the MHD problem. Unfortunately, the projection scheme in [12] does not allow for a discrete energy estimate. Moreover, it was proved that this scheme provided the weakly order $\frac{1}{2}$ approximations of the velocity field and the magnetic field in $\mathbf{H}^1(\Omega)$. In this paper, we propose a new linearized projection scheme, which allows for a discrete energy inequality, thus, and is unconditionally stable. We will prove that this new projection scheme provides the weakly first-order approximations of the velocity field and the magnetic field in $\mathbf{H}^1(\Omega)$, and the strongly first-order approximations of the velocity field and the magnetic field in $\mathbf{L}^2(\Omega)$ under the regularity Assumptions 1 and 2 below. To obtain the optimal convergence order of the pressure, we need some further regularity assumption as like in [16], under which we can derive the weakly first-order approximation of the pressure in $L^2(\Omega)$.

This paper is organized as follows: in the next section, we begin with some notation and lay out some regularity assumptions and recall some known inequalities frequently used. The new linearized projection scheme and the main results are presented in Section 3. Meanwhile, the discrete energy inequality is derived in Section 3. The proofs of main results are given in Sections 4 and 5, which are split into several theorems.