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A novel finite difference scheme for Burgers' equation on unbounded domains ☆

Quan Zheng *, Xin Zhao, Yufeng Liu

College of Sciences, North China University of Technology, Beijing 100144, China

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ABSTRACT

This paper studies a finite difference method for one-dimensional nonhomogeneous Burgers' equation on the infinite domain. Two exact nonlinear artificial boundary conditions are applied on two artificial boundaries to limit the original problem onto a bounded computational domain. A function transformation makes both Burgers' equation and artificial boundary conditions linear. Consequently, a novel finite difference scheme is developed by using the method of reduction of order for the obtained equation and artificial boundary conditions. The stability and the convergence with order 3/2 in time and 2 in space in an energy norm are proved for this method for Burgers' equation. Different examples illustrate the unconditional stability and the accuracy of the proposed method.

1. Introduction

Burgers' equation over an unbounded domain is a mathematical model employed to deal with various external problems of hydro- and aerodynamics, flows in porous media, turbulence in fluid dynamics and so on [1–3,14,16]. Burgers' equation is also frequently creamed off to design computational methods and appraise the stability and the efficiency of computational methods in order to tackle more difficult problems such as the Navier–Stokes equation. It can be linearized to heat equation by Hopf–Cole transformation and the resulting heat equation is solved by a large number of numerical methods on the finite domain [4,10–13].

It is a challenge to design effective numerical methods to overcome the unboundedness of the physical domains and make theoretical analyses to obtain the stability and the convergence of these methods for external problems of heat equation and Burgers' equation. On the one hand, Han and Huang (2002) in [5] derived the exact artificial boundary conditions, used FEM and FDM to solve three heat problems in one-dimensional cases, but had no error estimate. Han and Huang (2002) in [6] also obtained the exact artificial boundary condition and a series of approximating artificial boundary conditions, applied FEM and FDM for heat problems in two dimensions, and only showed its effectiveness by numerical examples, too. Wu and Sun (2004) in [17] investigated the FDM for the heat problem on the semi-infinite interval from [5], claimed proving the unique solvability, the unconditional stability and the convergence with the order 2 in space and the order 3/2 in time (i.e., $O(\tau^{3/2}+h^2)$) under an energy norm, but should have applied Gronwall inequality in their proof. On the other hand, the exact nonlinear artificial boundary condition for Burgers' equation was derived by the exact artificial boundary condition for the heat problem in the first one-dimensional case from [5], a nonlinear finite difference method was

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* Corresponding author.
E-mail address: zhengq@ncut.edu.cn (Q. Zheng).

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Table 1
Methods and their properties for heat/Burgers' equation on infinite domains.

Authors' paper	PDE and its spacial region	Numerical methods	Theoretical analysis	Their conclusions
H-H (2002) [5]	$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(x,t)$ $-1 \le x < +\infty$	FDM, FEM	–	feasible and effective
H-H (2002) [5]	$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} = f(r,t)$ $a \le r < +\infty$	FDM, FEM	–	feasible and effective
H-H (2002) [5]	$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} = f(r,t)$ $a \le r < +\infty$	FDM, FEM	–	feasible and effective
H-H (2002) [6]	$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial t} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ $a \le r < +\infty$	FDM, FEM	–	feasible and effective
W-S (2004) [17]	$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t)$ $-1 \le x < +\infty$	FDM	stability and $O(\tau^{\frac{3}{2}} + h^2)$	unconditionally stable convergent as $O(\tau^{\frac{3}{2}} + h^2)$
H-W-X (2006) [9]	$u_t + uu_x - \sigma u_{xx} = f(x,t)$ $-\infty < x < +\infty$	nonlinear FDM	–	very effective and converge fast
Z-F-L (2014) [21]	$u_t + uu_x - \sigma u_{xx} = 0$ $-\infty < x < +\infty$	linear FDM	stability and $O(\tau^{\frac{3}{2}} + h^2)$	unconditionally stable convergent as $O(\tau^{\frac{3}{2}} + h^2)$
S-W (2009) [15]	$u_t + uu_x - \sigma u_{xx} = f(x,t)$ $-\infty < x < +\infty$	linear FDM	stability and $O(\tau^{\frac{3}{2}} + h^2)$	unconditionally stable convergent as $O(\tau^{\frac{3}{2}} + h^2)$

constructed and its effectiveness was showed by numerical examples by Han, Wu and Xu (2006) in [9]. Burgers' equation on the unbounded domain was converted to the heat equation on the unbounded domain by Hopf–Cole transformation and confined to be the initial-boundary value problem by using the first exact artificial boundary condition from [5], the FDM for the resulting initial-boundary value problem was established, and the stability and the convergence $O(\tau^{3/2} + h^2)$ under an energy norm were proved by using some techniques from [17] by Zheng, Fan and Li (2014) in [21]. Furthermore, Burgers' equation and nonlinear artificial boundary conditions in [9] were reduced to be linear by a function transformation, a finite difference scheme was established by the method of reduction of order, and the stability and the convergence $O(\tau^{3/2} + h^2)$ were proved by using some techniques from [17] by Sun and Wu (2009) in [15]. The related methods and their properties for heat equation and Burgers' equation on infinite domains are summarized in Table 1.

Let us consider the following Cauchy problem of one-dimensional nonhomogeneous nonlinear viscous Burgers' equation:

$$u_t + uu_x - \sigma u_{xx} = f(x,t), \quad -\infty < x < +\infty, 0 < t \le T, \tag{1}$$

$$u(x,0) = \phi(x), \quad -\infty < x < +\infty, \tag{2}$$

$$u(x,t) \to 0, \quad \text{when } |x| \to +\infty, \quad 0 \le t \le T, \tag{3}$$

where σ is the kinematic viscosity parameter related to the Reynolds number, and the initial term $\phi(x)$ and the source term $f(x,t)$ have compact supports in space.

Introducing artificial boundaries $\Gamma_l = \{(x,t)|x = x_l, 0 \le t \le T\}$ and $\Gamma_r = \{(x,t)|x = x_r, 0 \le t \le T\}$, such that $\mathrm{supp}\{f(x,t)\} \subset [x_l, x_r] \times [0,T]$ and $\mathrm{supp}\{\phi(x)\} \subset [x_l, x_r]$, Han, Wu and Xu in [9] proposed two nonlinear exact artificial boundary conditions:

$$u(x_r,t) = -\frac{1}{\sqrt{\pi\sigma}} \int_0^t \frac{e^{g_r(\lambda)-g_r(t)}}{\sqrt{t-\lambda}} (\sigma u_x(x_r,\lambda) - \frac{u(x_r,\lambda)^2}{2}) d\lambda,$$

$$u(x_l,t) = \frac{1}{\sqrt{\pi\sigma}} \int_0^t \frac{e^{g_l(\lambda)-g_l(t)}}{\sqrt{t-\lambda}} (\sigma u_x(x_l,\lambda) - \frac{u(x_l,\lambda)^2}{2}) d\lambda,$$

where

$$g_r(t) = \int_0^t -\frac{u_x(x_r,\lambda)}{2} + \frac{u(x_r,\lambda)^2}{4\sigma} d\lambda, \quad g_l(t) = \int_0^t -\frac{u_x(x_l,\lambda)}{2} + \frac{u(x_l,\lambda)^2}{4\sigma} d\lambda.$$

Sun and Wu used in [15] the above artificial boundary conditions and a function transformation as follows. Integrate (1) for x over $\mathbb{R}$, then

$$\frac{d}{dt} \int_{-\infty}^{+\infty} u(y,t) dy = \int_{-\infty}^{+\infty} f(y,t) dy = \int_{x_l}^{x_r} f(y,t) dy \equiv \psi(t).$$