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An adaptive meshless local Petrov–Galerkin method based on a posteriori error estimation for the boundary layer problems

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ABSTRACT

A new adaptive moving least squares (MLS) method with variable radius of influence is presented to improve the accuracy of Meshless Local Petrov–Galerkin (MLPG) methods and to minimize the computational cost for the numerical solution of singularly perturbed boundary value problems. An error indicator based on a posteriori error estimation, accurately captures the regions of the domain with insufficient resolution and adaptively determines the new nodes location. The effectiveness of the new method is demonstrated on some singularly perturbed problems involving boundary layers.

1. Introduction

Adaptive numerical methods are essential tools to improve the accuracy of numerical approximations to many important physical problems. These techniques have become increasingly important because of their great capacity for localizing when the total accuracy of numerical methods destroys due to interior or boundary layers, local singularities arisen from re-entrant corners and sharp moving fronts [6]. Although adaptive finite element techniques have achieved great success, they have faced difficulties with large mesh distortion when dealing with large deformation problems [26,28].

In the last few decades, a new class of numerical methods, meshless methods, has been developed to overcome or at least to reduce the complexity and the computational cost of meshing the complex structural domains [21–23,29]. Meshless methods are generally formulated based on a set of scattered nodes. This significant feature also simplifies meshless methods to equip with adaptive techniques to perform adaptive analysis, because the nodes can be removed or inserted easily.

Meshless local Petrov–Galerkin method proposed by Atluri and Zhu [4], is one of the most frequently utilized and developed meshless methods. It is a truly meshless method which is based on a set of local weak forms, rather than a single global weak form. All integrations are always performed over regularly shaped and overlapping sub-domains Ω_j that cover the problem domain Ω. Therefore no background cells are required in either the approximation or the integration.

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This method is believed to have a good future due to its general form of test and trial functions. Most of the meshless methods can be derived from it, as special cases, based on which of the possible trial and test functions or integration method are chosen. The most frequently used trial and test functions are MLS, radial basis functions (RBFs), partition of unity (PU) and Shepard function [1].

As Atluri and Shen classified in [1], based on the concept of the MLPG, the test functions over each local Ω_j can be chosen through a variety of ways. The test function over Ω_j can be the same as the weight function in the MLS approximation (MLPG1), collocation Dirac's Delta function over Ω_j (MLPG2), discrete least squares error function in the differential equation (MLPG3), the modified fundamental solution to the differential equation (MLPG4), the Heaviside step function over Ω_j (MLPG5), and the test function can be the same as the trial function, i.e. the Galerkin method (MLPG6) [1]. MLPG2 method is also called the finite point method (FPM) [35] when moving least squares approximation is applied as the trial function. Also MLPG4 is called Local Boundary Integral Equation (LBIE) [45]. The standard MLPG formulation relying on MLS approximation can be seen in some references, [1–3,8–11,32,43–46,48,49].

Recently, a new MLPG method has been proposed, called direct MLPG or DMLPG [33], based on the generalized moving least squares approximation [34]. In contrast to the MLPG method, GMLS directly recovers the test functionals directly from values at nodes. This follows the main feature of meshless methods to express all the values in terms of nodes [33]. This leads to a less costly scheme which shifts the local integrations over low-degree polynomials instead of complicated shape functions. In the other words, MLPG generalizes finite element methods, while DMLPG generalizes finite difference techniques. We refer the interested reader to [33,38,47] for more details on analysis and implementation of this method.

Since meshless methods are based generally on a set of quasi-regular approximation nodes to define the basis functions, locally refining and coarsening nodes are straightforward for these methods [13]. In the recent years several researchers have introduced several adaptive RBF schemes for both time-independent and time-dependent problems. Schaback and Wendland [41] and Hon et al. [17] proposed adaptive methods based on a greedy algorithm and best n-term approximation using compactly supported RBFs for interpolation and collocation problems. Authors of [17] introduced an indicator based on the weak formulation of the governing equation to adaptively re-allocate more RBF approximation points to the boundary layer. Ling and Trummer [27] developed a robust adaptive scheme based on a multiquadric integral formulation to solve problems with extremely thin boundary layers.

In [40] Sarra applied a modification of a simple moving grid algorithm, which was developed for use with low order finite difference method, to RBF methods for solving time dependent PDEs. The method is essentially the method of lines with RBFs implementation of the uniform distribution of arc length algorithm in space [40]. Driscoll and Heryudono in [13], proposed a new method for adaptive RBF based on residual subsampling for radial basis function in interpolation and collocation problems.

Also several researches have been done on adaptive MLS and its application in surface reconstructions in computer graphics. Huang et al. [18,19] presented an adaptive algorithm based on the moving least squares methods (MLS) for non-uniform sampled points fitting. Dey and Sun in [12] proposed a new variant of the MLS surface that considers local feature sizes in its formulation and guarantees using a non-uniform sampling density. They claim that the proposed method can be used effectively to reconstruct surfaces from possibly noisy point cloud data [12]. Pudjisuryadi [36] presented an adaptive MLPG method using Lagrange interpolation functions as test functions with polygonal sub-domains constructed from several triangular patches rather than the typically used circular sub-domains.

The purpose of this paper is to present a new adaptive MLS approximation and extend it to the adaptive MLPG method. The adaptive MLS procedure, based on residual subsampling is explained in Section 2. We also demonstrate the effectiveness of this technique for approximation in 1D and 2D regions. In Section 3, an adaptive MLPG and a posteriori error estimation are proposed. We demonstrate that this method is capable of achieving high accuracy through the adaptivity for problems with boundary layers in Section 4. Finally, some concluding remarks are given in Section 5.

2. Adaptive MLS approximation

The moving least squares approximation was introduced by Shepard [42] in the late 1960s for constructing smooth approximations to fit a specified cloud of points. It was then extended in [24] for general surface generation problems. In what follows we briefly describe the concept of MLS approximation.

2.1. MLS approximation

The MLS approximation $\tilde{u} : \Omega \to \mathbb{R}$ of the function $u : \Omega \to \mathbb{R}$ is posed as a polynomial of order m but with non-constant coefficients. Let us begin with the problem of approximating a function $u(\mathbf{x})$ in domain Ω, over a number of randomly located nodes $\{\mathbf{x}_i\}$, $i = 1, 2, ..., N$. The MLS approximation $\tilde{u} : \Omega \to \mathbb{R}$ of the function $u : \Omega \to \mathbb{R}$ is defined by

$$\tilde{u}(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}), \qquad \forall \mathbf{x} \in \bar{\Omega}, \tag{2.1}$$

where $\mathbf{p}^T(\mathbf{x}) = \big(p_1(\mathbf{x}), p_2(\mathbf{x}), ..., p_l(\mathbf{x})\big)$ is the basis vector of polynomials of order m. For example in the two dimensional case, the basis function $\mathbf{p}^T(\mathbf{x})$ is given by

Linear basis $(m = 1,\ l = 3)$ $\mathbf{p}^T(\mathbf{x}) = \big(1, x, y\big)$,