Applied Numerical Mathematics 102 (2016) 1–16

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Benchmark results for testing adaptive finite element eigenvalue procedures part 2 (conforming eigenvector and eigenvalue estimates)

Stefano Giani [a], Luka Grubišić [b,*], Jeffrey S. Ovall [c]

[a] Durham University, School of Engineering and Computing Sciences, South Road, Durham DH1 3LE, UK
[b] University of Zagreb, Department of Mathematics, Bijenička 30, 10000 Zagreb, Croatia
[c] Portland State University, Fariborz Maseeh Department of Mathematics and Statistics, 315 Neuberger Hall, Portland, OR 97201, USA

ARTICLE INFO

Article history:
Received 16 May 2014
Received in revised form 2 June 2015
Accepted 21 December 2015
Available online 31 December 2015

Keywords:
Eigenvalue problems
Finite element methods
A posteriori error estimates
hp-Adaptivity

ABSTRACT

We present an hp-adaptive continuous Galerkin (hp-CG) method for approximating eigenvalues of elliptic operators, and demonstrate its utility on a collection of benchmark problems having features seen in many important practical applications—for example, high-contrast discontinuous coefficients giving rise to eigenfunctions with reduced regularity. In this continuation of our benchmark study, we concentrate on providing reliability estimates for assessing eigenfunction/invariant subspace error. In particular, we use these estimates to justify the observed robustness of eigenvalue error estimates in the presence of repeated or clustered eigenvalues. We also indicate a means for obtaining efficiency estimates from the available efficiency estimates for the associated boundary value (source) problem. As in the first part of the paper we provide extensive numerical tests for comparison with other high-order methods and also extend the list of analyzed benchmark problems.

1. Introduction

In Part 1 of this work [14], we presented an hp-adaptive discontinuous Galerkin (hp-DG) method for approximating eigenvalues of elliptic operators, and demonstrated its utility on a collection of benchmark problems having features seen in many important practical applications—for example, high-contrast discontinuous coefficients giving rise to eigenfunctions with reduced regularity. This approach was shown to be highly efficient in terms of cost per correct digit, and provided computed error estimates which were asymptotically identical to the actual errors. Our DG work did not, however, provide a means of assessing eigenfunction/invariant subspace error; nor did it justify the observed robustness of eigenvalue error estimates in the presence of repeated eigenvalues. In this continuation of our benchmark study, we present a continuous Galerkin hp-adaptive (hp-CG) method which addresses these issues. As in Part 1, we provide extensive numerical tests for comparison with other high-order methods. Our CG work builds on the abstract framework of [20,7] for eigenvalue/invariant subspace error estimation, as outlined in Section 2, providing a more robust theory than was possible in our previous approach. Concerning *hp*-adaptive approximation methods for eigenvalue problems, much more has been written about their *a priori* error analysis than their *a posteriori* error analysis, and we mention Refs. [5,24,29], as well as the more comprehensive

* Corresponding author.
E-mail addresses: stefano.giani@nottingham.ac.uk (S. Giani), luka.grubisic@math.hr (L. Grubišić), jovall@pdx.edu (J.S. Ovall).

http://dx.doi.org/10.1016/j.apnum.2015.12.001
0168-9274/

general survey [11], as recent references in this vein. A recent contribution to *a posteriori* analysis which is most readily compared with our own is that of [4], and we make relevant comparisons and contrasts with this work in Sections 3 and 4.

Our model problem is as follows. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, and let $\partial\Omega_D \subset \partial\Omega$ have positive (1D) Lebesgue measure. We define the space $\mathcal{H} = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$, where these boundary values are understood in the sense of trace. We are interested in the eigenvalue problem:

$$\text{Find } (\lambda, \psi) \in \mathbb{R} \times \mathcal{H} \text{ so that } B(\psi, v) = \lambda(\psi, v) \text{ and } \psi \neq 0 \text{ for all } v \in \mathcal{H}, \tag{1}$$

where

$$B(w, v) = \int_\Omega A\nabla w \cdot \nabla v + cwv\, dx, \qquad (w, v) = \int_\Omega wv\, dx. \tag{2}$$

We assume that the diffusion matrix A is piecewise constant and uniformly positive definite a.e., and the scalar c is also piecewise constant and non-negative. The assumption that A and c are piecewise constant, as opposed to just being piecewise smooth, is a theoretical convenience, as is the assumption that Ω is a polygon. These are not practical limitations to our approach. The discontinuities in these coefficients permit investigation of many interesting eigenvalue model problems for composite materials, such as those which are of interest for methods of nondestructive sensing (cf. [1,2]). Our interest in problems of this sort is motivated by considerations of photonic crystals and related problems (cf. [3,13]). Such applications are not directly addressed in this work, but many of their inherent computational challenges are captured in our model problem.

The assumptions on the coefficients A and c are sufficient to guarantee that the bilinear form is both bounded and coercive on $H^1(\Omega)$,

$$B(v, w) \leq \beta_1 \|v\|_1 \|w\|_1, \qquad B(v, v) \geq \beta_0 \|v\|_1^2 \text{ for all } v, w \in \mathcal{H},$$

for some constants $\beta_0, \beta_1 > 0$. So $B(\cdot, \cdot)$ is an inner product on $\mathcal{H}$, whose induced energy-norm $|||\cdot|||$ is equivalent to $\|\cdot\|_1$. Here and elsewhere, $|\cdot|_1$ and $\|\cdot\|_1$ denote the usual semi-norm and norm on $H^1(\Omega)$, and $\|\cdot\|_0$ denotes the norm on $L^2(\Omega)$. For some results, it is convenient to restrict the norms/semi-norms to a subset $S \subset \Omega$ (in the obvious way), and we denote these restrictions by $\|\cdot\|_{0,S}$, $|\cdot|_{1,S}$, $\|\cdot\|_{1,S}$ and $|||\cdot|||_S$. The we point out that $|||\cdot|||_S$ may actually be a semi-norm, though our assumptions on the coefficients A and c guarantee that there are local constants $\beta_{0S}, \beta_{1S} > 0$ such that $\beta_{0S}|v|_{1,S}^2 \leq |||v|||_S^2 \leq \beta_{1S}\|v\|_{1,S}^2$, and the seminorm in the lower bound can be replaced with the full norm (after modifying β_{0S} if necessary) if $c(x) \geq c_S > 0$ on S.

The variational eigenvalue problem (1) is satisfied by the positive sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_q \leq \cdots \tag{3}$$

and a sequence of eigenvectors $(\psi_i)_{i\in\mathbb{N}}$,

$$B(\psi_i, v) = \lambda_i(\psi_i, v) \text{ for all } v \in \mathcal{H}, \qquad \text{and } (\psi_i, \psi_j) = \delta_{ij}. \tag{4}$$

Here we have counted the eigenvalues according to their multiplicity. Furthermore, the sequence $(\lambda_i)_{i\in\mathbb{N}}$ has no finite accumulation points. For path-wise connected domains Ω, we know via the Peron–Frobenius Theorem, that $\lambda_1 < \lambda_2$ holds and that ψ_1 can be chosen to be continuous and strictly positive in Ω. We will also use the notation

$$\text{Spec}_B = \{\lambda_i : i \in \mathbb{N}\}, \qquad \mathfrak{M}(\lambda) = \text{span}\{\psi \in \mathcal{H} : B(\psi, \phi) = \lambda(\psi, \phi) \text{ for all } \phi \in \mathcal{H}\}$$

to denote the spectrum of the variational eigenvalue problem, and the spectral subspace (invariant subspace) associated to $\lambda \in \text{Spec}_B$, noting that these subspaces are finite dimensional in our setting. Furthermore, let E_λ be the L^2-orthogonal projection onto $\mathfrak{M}(\lambda)$. Then

$$\sum_{\lambda\in\text{Spec}_B} E_\lambda = I$$

and the spaces $\mathfrak{M}(\lambda) = \text{Ran}\, E_\lambda$ and $\mathfrak{M}(\mu) = \text{Ran}\, E_\mu$ are mutually orthogonal for distinct $\lambda, \mu \in \text{Spec}_B$. We finally note that

$$B(\psi, \phi) = \sum_{\lambda\in\text{Spec(A)}} \lambda(\psi, E_\lambda\phi), \qquad \psi, \phi \in \mathcal{H}$$

and so we obtain an alternative representation of the energy norm

$$|||\psi|||^2 = B(\psi, \psi) = \sum_{\lambda\in\text{Spec(A)}} \lambda(\psi, E_\lambda\psi), \quad \psi \in \mathcal{H}. \tag{5}$$

The rest of this paper is organized as follows. In Section 2 we introduce basic notation related to the *hp*-discretization, and discuss *approximation defects* as ideal error estimates, with key results from [20,7] extended for use in the present context. These extensions make possible the incorporation of results from [26, Section 3] to obtain efficient and reliable