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H-kernels by walks in H-colored digraphs and the color-class digraph

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Abstract

Let H be a digraph possibly with loops and D a finite digraph without loops whose arcs are colored with the vertices of H (D is an H-colored digraph). $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. For an arc (z_1, z_2) of D we will denote by $c_D(z_1, z_2)$ its color. A directed walk (respectively directed path) $(v_1, v_2, \ldots, v_n)$ in D is an H-walk (respectively H-path) if and only if $(c_D(v_1, v_2), c_D(v_2, v_3), \ldots, c_D(v_{n-1}, v_n))$ is a directed walk in H. A set $K \subseteq V(D)$ is an H-kernel by walks (respectively H-kernel) if for every pair of different vertices in K there is no H-walk (respectively H-path) between them, and for every vertex $u \in V(D) \setminus K$ there exists $v \in K$ such that there exists an H-walk (respectively H-path) from u to v in D.

Let D be an arc-colored digraph. The color-class digraph of D, denoted by $\mathscr{C}_C(D)$, is defined as follows: the vertices of the color-class digraph are the colors represented in the arcs of D and $(i, j) \in A(\mathscr{C}_C(D))$ if and only if there exist two arcs namely $(u, v) \in A(D)$ colored i and $(v, w) \in A(D)$ colored j. In this paper we relate the concepts discussed above, the color-class digraph and the H-coloration of D, in order to prove the existence of an H-kernel by walks (respectively H-kernel).

Keywords: H-kernel; Kernel; Kernel by monochromatic paths; Color-class digraph

1. Introduction

For general concepts we refer the reader to [1] and [2]. A *directed walk* is a sequence $(v_1, v_2, \ldots, v_n)$ such that $(v_i, v_{i+1}) \in A(D)$ for each $i \in \{1, \ldots, n-1\}$. Moreover if $v_i \neq v_j$ for $i \neq j, \{i, j\} \subseteq \{1, \ldots, n\}$, then it is called *directed path*. A *directed cycle* is a directed walk $(v_1, v_2, \ldots, v_n, v_1)$ such that $v_i \neq v_j$ for $i \neq j, \{i, j\} \subseteq \{1, \ldots, n\}$. If D is an infinite digraph, an *infinite outward path* is an infinite sequence $(v_1, v_2, \ldots)$ of distinct vertices of D such that $(v_i, v_{i+1}) \in A(D)$ for each $i \in N$. In this paper we are going to write walk, path, cycle instead of directed walk, directed path, directed cycle, respectively.

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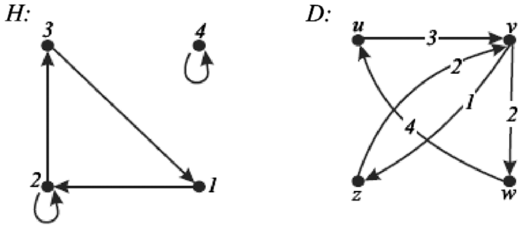

Fig. 1. $\{w\}$ is an H-kernel by walks of D and D has no H-kernel.

A digraph D is said to be *arc-colored* if its arcs are colored. A digraph D is said to be m-colored if the arcs of D are colored with m colors. Let D be an arc-colored digraph. A path is called *monochromatic* if all of its arcs are colored alike. For an arc (z_1, z_2) of D we will denote by $c_D(z_1, z_2)$ its color.

In [3] Sands et al. proved that if the arcs of a finite tournament are colored with two colors, then there is always a single vertex reachable from any other by a monochromatic path. In [4] Linek and Sands gave an extension of the result of Sands et al. in which the arcs of a tournament T are colored with the elements of a partially ordered set P. They called a path $(v_1, \ldots, v_n)$ in T monotone if $c_T(v_i, v_{i+1}) \leq c_T(v_{i+1}, v_{i+2})$ in P for each i. In [4] Linek and Sands considered a further extension as follows: if H is a reflexive digraph and T is a tournament whose arcs are colored by the vertices of H, an H-path W in T is a path in T for which $(c_T(u, v), c_T(v, w)) \in \mathrm{A}(H)$ for any two consecutive arcs (u, v) and (v, w) in W.

In [5] Arpin and Linek reconsidered the last extension suggested in [4] in order to assign a color to the arcs of a multidigraph D with the vertices of a digraph H (possibly irreflexive). They called a walk or a path $(v_1, v_2, \ldots, v_n)$ in D an H-walk or an H-path, respectively, iff $(c_D(v_1, v_2), c_D(v_2, v_3), \ldots, c_D(v_{n-1}, v_n))$ is a walk in H. Notice that an arc is an H-path, that is to say, a singleton vertex is a walk in H. They also called a set of vertices $S \subseteq \mathrm{V}(D)$ H-absorbent by walks if for every $x \in \mathrm{V}(D) \setminus S$ there is an H-walk from x to some point of S and a set $I \subseteq \mathrm{V}(D)$ was called H-independent by walks if there is no H-walk between any two distinct vertices of I. Since the existence of an H-walk between two vertices does not guarantee the existence of an H-path between those vertices (although for some H this is true) and the concatenation of two H-paths is not always an H-path, in [5] Arpin and Linek prefer to work with H-walks instead of H-paths. In [5] they classify $\mathscr{B}_2$ (the class of all H such that any multidigraph D arc-colored with the vertices of H has an independent set of vertices that is H-absorbent by walks) and they make inroads in the classification of $\mathscr{B}_3$ (the class of all H such that any multidigraph D arc-colored with the vertices of H has a set of vertices S that is both H-independent by walks and H-absorbent by walks) and $\mathscr{B}_1$ (the class of all H such that any tournament arc-colored with the vertices of H has a single vertex H-absorbent by walks).

In [6] Galeana-Sánchez and Delgado-Escalante used the work of Arpin and Linek [5] in order to introduce the following concepts:

Definition 1.1. A subset N of $\mathrm{V}(D)$ is said to be an H-kernel by walks if it satisfies the following two conditions:

1. For every pair of different vertices in N there is no H-walk between them in D (N is H-independent by walks in D).
2. For every vertex y in $\mathrm{V}(D) \setminus N$ there is an H-walk from y to N in D (N is H-absorbent by walks in D).

Definition 1.2. A subset N of $\mathrm{V}(D)$ is said to be an H-kernel if it satisfies the following two conditions:

1. For every pair of different vertices in N there is no H-path between them in D (N is H-independent in D).
2. For every vertex y in $\mathrm{V}(D) \setminus N$ there is an H-path from y to a vertex in N (N is H-absorbent in D).

Since the existence of an H-walk between two vertices does not guarantee the existence of an H-path between those vertices and the concatenation of two H-paths is not always an H-path, we can claim that if D has an H-kernel by walks, then D not necessarily has an H-kernel as the example in Fig. 1 shows. In Fig. 1 we have that $\{w\}$ is an H-kernel by walks of D, because (u, v, z, v, w) is an H-walk in D that finishes in w and it contains every vertex of D. It is easy to check that D has no H-kernel (notice that every H-independent set of D has cardinality one).

We also claim that if D has an H-kernel, then D not necessarily has an H-kernel by walks as the example in Fig. 2 shows. In Fig. 2 we have that $\{u, x\}$ is an H-kernel in D. It is easy to see that D has no H-kernel by walks (notice