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Graphs with vertex-coloring and detectable 2-edge-weighting

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Abstract

For a connected graph G of order $|V(G)| \geq 3$ and a k-edge-weighting $c : E(G) \rightarrow \{1, 2, \ldots, k\}$ of the edges of G, the *code*, $code_c(v)$, of a vertex v of G is the ordered k-tuple $(\ell_1, \ell_2, \ldots, \ell_k)$, where ℓ_i is the number of edges incident with v that are weighted i. (i) The k-edge-weighting c is *detectable* if every two adjacent vertices of G have distinct codes. The minimum positive integer k for which G has a detectable k-edge-weighting is *the detectable chromatic number* $det(G)$ of G. (ii) The k-edge-weighting c is a *vertex-coloring* if every two adjacent vertices u, v of G with codes $code_c(u) = (\ell_1, \ell_2, \ldots, \ell_k)$ and $code_c(v) = (\ell'_1, \ell'_2, \ldots, \ell'_k)$ have $1\ell_1 + 2\ell_2 + \cdots + k\ell_k \neq 1\ell'_1 + 2\ell'_2 + \cdots + k\ell'_k$. The minimum positive integer k for which G has a vertex-coloring k-edge-weighting is denoted by $\mu(G)$. In this paper, we have enlarged the known families of graphs with $det(G) = \mu(G) = 2$.

Keywords: Detectable edge-weighting; Vertex-coloring edge-weighting; Cartesian product; Tensor product

1. Introduction

For graph-theoretical terminology and notation, we in general follow [1]. In this paper, we assume that the graphs G in discussion are finite, connected, undirected and simple with order $|V(G)| \geq 3$.

Let $c : E(G) \rightarrow \{1, 2, \ldots, k\}$ be a k-edge-weighting of G, where k is a positive integer. The *color code* of a vertex v of G is the ordered k-tuple $code_c(v) = (\ell_1, \ell_2, \ldots, \ell_k)$, where ℓ_i is the number of edges incident with v that are weighted i for $i \in \{1, 2, \ldots, k\}$. Therefore, $\ell_1 + \ell_2 + \cdots + \ell_k = d_G(v)$, the degree of v in G. It follows that for $u, v \in V(G)$ if $d_G(u) \neq d_G(v)$, then $code_c(u) \neq code_c(v)$. The k-edge-weighting c of G is called *detectable* if every two adjacent vertices of G have distinct color codes. The *detectable chromatic number* $det(G)$ of G is the minimum positive integer k for which G has a detectable k-edge-weighting.

Any k-edge-weighting $c : E(G) \rightarrow \{1, 2, \ldots, k\}$ induces a vertex-weighting $f_c : V(G) \rightarrow \mathbb{N}$ defined by $f_c(v) = \sum_{e \text{ is incident with } v} c(e)$. An edge-weighting c is a *vertex-coloring* if $f_c(u) \neq f_c(v)$ for any edge uv. Denote by $\mu(G)$ the minimum k for which G has a vertex-coloring k-edge-weighting.

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If a graph has an edge as a component, then it neither has a detectable edge-weighting nor has a vertex-coloring edge-weighting. So in this paper, we only consider graphs without a K_2 component and such graphs are called *nice graphs*. As the graph G in discussion is connected and as $|V(G)| \geq 3$, G is nice.

Karoński et al. [2] initiated the study of vertex-coloring k-edge-weighting and they posed the following conjecture:

Conjecture 1.1 (*1-2-3-Conjecture*). *Every nice graph admits a vertex-coloring* 3*-edge-weighting.*

Consider a vertex-coloring k-edge-weighting c of G. For $uv \in E(G)$, let ℓ_i, ℓ'_i, respectively, be the number of edges incident with u, v that are weighted i in c. Then $1\ell_1 + 2\ell_2 + \cdots + k\ell_k \neq 1\ell'_1 + 2\ell'_2 + \cdots + k\ell'_k$ and hence $(\ell_1, \ell_2, \ldots, \ell_k) \neq (\ell'_1, \ell'_2, \ldots, \ell'_k)$. So c is a detectable k-edge-weighting. Consequently, $det(G) \leq \mu(G)$.

Proposition 1.1. $det(G) \leq \mu(G)$.

Proposition 1.2. *For every nice graph G, following three conditions are equivalent:*
(i) $det(G) = 1$,
(ii) $\mu(G) = 1$,
(iii) *G has no adjacent vertices with the same degree.*

Proposition 1.3. *If* $\mu(G) = 2$*, then* $det(G) = 2$.

If c is a detectable 2-edge-weighting of a k-regular graph G with $k \geq 3$, then c is a vertex-coloring 2-edge-weighting. This follows from the fact that $\ell_1 + \ell_2 = k = \ell'_1 + \ell'_2$ and $(\ell_1, \ell_2) \neq (\ell'_1, \ell'_2)$ imply $1\ell_1 + 2\ell_2 \neq 1\ell'_1 + 2\ell'_2$.

Proposition 1.4. *Let G be a k-regular graph with* $k \geq 3$*. If* $det(G) = 2$*, then* $\mu(G) = 2$.

In [2], Karoński et al. proved that: (i) $det(G) \leq 183$, and (ii) if $d_G(v) \geq 10^{99}$ for every $v \in V(G)$, then $det(G) \leq 30$.

In [3], Addario-Berry et al. proved that: (i) $det(G) \leq 4$, (ii) if $d_G(v) \geq 1000$ for every $v \in V(G)$, then $det(G) \leq 3$, and (iii) if $\chi(G) \leq 3$, then $det(G) \leq 3$.

In [4], among other results, Escuadro et al. proved that: (i) $det(K_{n_1,n_2,\ldots,n_k}) = 1$ if $n_1 < n_2 < \cdots < n_k$, $det(K_{n_1,n_2,\ldots,n_k}) = 3$ if $n_1 = n_2 = \cdots = n_k = 1$ and $det(K_{n_1,n_2,\ldots,n_k}) = 2$ otherwise, where $K_{n_1,n_2,\ldots,n_k}$ is the complete k-partite graph with partite sizes $n_1, n_2, \ldots, n_k$ ($k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$), (ii) $det(C_3 \square K_2) = 3$, $det(C_5 \square K_2) = 3$ and if $n \geq 7$ is an odd integer, then $det(C_n \square K_2) = 2$, where $\square$ denotes the Cartesian product, and (iii) if G is a unicyclic graph that is not a cycle, then $det(G) \leq 2$.

See Fig. 5 of [4]; detectable 3-edge-weighting of $C_3 \square K_2$ and that of $C_5 \square K_2$, in the figure, are vertex-coloring 3-edge-weightings. Hence, $\mu(C_3 \square K_2) = 3$ and $\mu(C_5 \square K_2) = 3$. If $n \geq 7$ is an odd integer, then it follows from $det(C_n \square K_2) = 2$ and Proposition 1.4 that $\mu(C_n \square K_2) = 2$.

Theorem 1.1. $det(C_3 \square K_2) = \mu(C_3 \square K_2) = 3$, $det(C_5 \square K_2) = \mu(C_5 \square K_2) = 3$ *and if* $n \geq 7$ *is an odd integer, then* $det(C_n \square K_2) = \mu(C_n \square K_2) = 2$.

From [5,6], and [4], we have:

Theorem 1.2. *For the path* P_n *on n vertices,* $det(P_3) = \mu(P_3) = 1$ *and* $det(P_n) = \mu(P_n) = 2$ *if* $n \geq 4$.

Theorem 1.3. *For the cycle* C_n *on n vertices,* $det(C_n) = \mu(C_n) = 2$ *if* $n \equiv 0\,(mod\,4)$ *and* $det(C_n) = \mu(C_n) = 3$ *if* $n \equiv 1,\ 2$ *or* $3\,(mod\,4)$.

Theorem 1.4. *For the complete graph* K_n *on* $n \geq 3$ *vertices,* $det(K_n) = \mu(K_n) = 3$.

Theorem 1.5. *For* $r + s \geq 3$, $det(K_{r,s}) = \mu(K_{r,s}) = 1$ *if* $r \neq s$ *and* $det(K_{r,s}) = \mu(K_{r,s}) = 2$ *if* $r = s$*, where* $K_{r,s}$ *is the complete bipartite graph with partite sizes r and s.*

The *theta graph* $\theta(\ell_1, \ell_2, \ldots, \ell_r)$ is the graph obtained from r disjoint paths $P_1(u_1, v_1), P_2(u_2, v_2), \ldots, P_r(u_r, v_r)$ of lengths $\ell_1, \ell_2, \ldots, \ell_r$, respectively, by identifying their end-vertices $u := u_1 = u_2 = \cdots = u_r$ and $v := v_1 = v_2 = \cdots = v_r$, where $P_i(u_i, v_i)$ is a path of length ℓ_i with origin u_i and terminus v_i. Note that $\theta(\ell_1) = P_{\ell_1+1}$ and $\theta(\ell_1, \ell_2) = C_{\ell_1+\ell_2}$.