Available online at www.sciencedirect.com
ScienceDirect
AKCE International Journal of Graphs and Combinatorics 13 (2016) 1–10
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

Puzzling and apuzzling graphs[☆]

Daphne Gold[a], James Henle[b,*], Cherry Huang[b], Tia Lyve[c], Tara Marin[b], Jasmine Osorio[d], Mäneka Puligandla[e], Bayla Weick[f], Jing Xia[b], He Yun[b], Jize Zhang[g]

[a] *Seattle, WA 98101, United States*
[b] *Smith College, Northampton, MA 01063, United States*
[c] *Ames, IA 50010, United States*
[d] *Lowe's, Charlotte, NC 28269, United States*
[e] *Dana Farber Institute, Boston, MA 02115, United States*
[f] *Palantir Technologies, New York, NY 10011, United States*
[g] *University of Washington, Seattle, WA 98195, United States*

Received 13 November 2014; received in revised form 2 January 2016; accepted 17 February 2016
Available online 9 March 2016

Abstract

Let G be a graph with chromatic number $\chi(G)$ and consider a partition P of G into connected subgraphs. P is a **puzzle** on G if there is a unique vertex coloring of G using $1, 2, \ldots, \chi(G)$ such that the sums of the numbers assigned to the partition pieces are all the same. P is an **apuzzle** if there is a unique vertex coloring such that the sums are all different.

We investigate the concept of puzzling and apuzzling graphs, detailing classes of graphs that are puzzling, apuzzling and neither.

Keywords: Vertex coloring; Puzzles; Partitions

1. Introduction

This research grew out of earlier work on variants of sudoku [1,2]. Instead of the usual partition of a square, the sudoku board was divided into different regions. No numbers were given, but the challenge was to place numbers so that no number appeared twice in a row or column, and the sums of the numbers in each region were the same. From puzzles such as these, it is only a few steps to puzzling graphs.

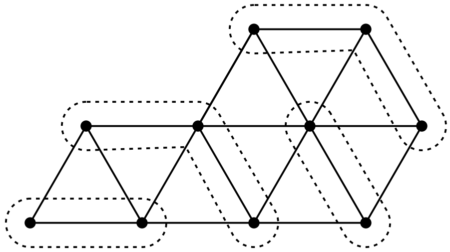
By "vertex coloring", we mean a minimal coloring of the vertices using the numbers $1, 2, \ldots, \chi(G)$. For this paper, "partition" will mean a partition where the pieces are all connected.

Peer review under responsibility of Kalasalingam University.

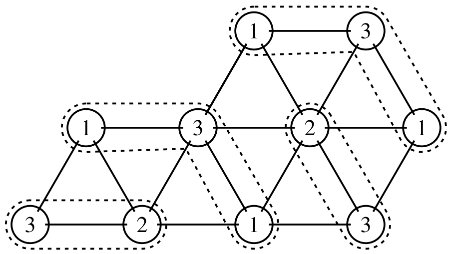
☆ This work was partially supported by NSF grant DMS 0611020 through the Center for Women in Mathematics at Smith College. In addition, the authors would like to thank Ruth Haas, Joan Hutchinson, and the referee for many helpful comments.
* Corresponding author.
E-mail address: jhenle@smith.edu (J. Henle).

http://dx.doi.org/10.1016/j.akcej.2016.02.001

Definition 1. Let G be a connected graph with chromatic number $\chi(G)$. A **puzzle** on G is a partition



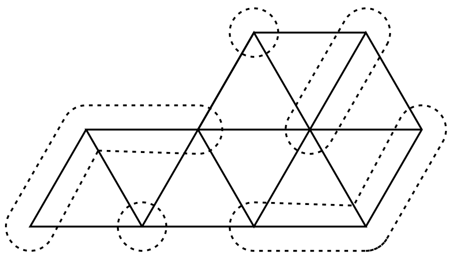
such that there is exactly one vertex coloring with the property that the sums of the vertex labels of the partition pieces are all the same.



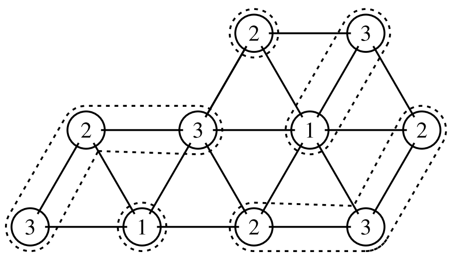
A graph is **puzzling** if there is a puzzle on the graph.

In [1] and [2], a second sort of puzzle made an appearance. Here the challenge was to place numbers so that the sums of the numbers in the regions were all different.

Definition 2. An **apuzzle** on a graph G is a partition of G



such that there is exactly one vertex coloring such that the sums of the vertex labels of the partition pieces are all different.



A graph is **apuzzling** if there is an apuzzle on the graph.

Many graphs are neither puzzling nor apuzzling. It is easy to see, for example, that for $n > 1$, K_n, the complete graph on n vertices, is not puzzling. A puzzle must have a piece with at least two vertices. Any solution to the puzzle generates another solution by switching the colors of the two vertices in the piece. A similar argument shows that K_n is not apuzzling.