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Some network topological notions of the Mycielskian of a graph

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Abstract

Efficiency and reliability are two important criteria in the designing of a good interconnection network. Network topological notions such as wide diameter, fault diameter, diameter vulnerability and (l, k)-domination can be used to study the efficiency and reliability of a network. In this paper we study these notions in the Mycielskian of a graph and its iterates.

Keywords: Mycielskian; Wide diameter; Diameter vulnerability; Fault diameter; (l, k)-domination

1. Introduction

An interconnection network connects the processors of a parallel and distributed system. The topological structure of an interconnection network can be modeled by a connected graph where the vertices represent components of the network and the edges represent communication links between them. Some graph theoretic techniques that are used to study the efficiency and reliability of a network are discussed in [1,2].

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The *degree* of a vertex u in G, $d(u)$ is the number of edges incident with u in G. The minimum degree and the maximum degree in a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The *neighborhood of* u, written $N(u)$, is the set $\{x \in V(G) : x \text{ is adjacent to } u\}$ and x is called a neighbor of u if $x \in N(u)$. The *subgraph of* G *induced by* $S \subseteq V(G)$ is denoted by $\langle S \rangle$.

The *distance* between u and v in a connected graph G, $d(u, v)$, is the length of a shortest path joining u and v in G. The *diameter* of a graph G, diam(G), is the maximum distance between any two vertices in G. The diameter is often taken as a measure of efficiency, especially for networks with maximum time-delay or signal degradation.

The *vertex connectivity*, $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal from G results in a disconnected graph or K_1. The *edge connectivity* of a connected graph G, $\kappa'(G)$ is the minimum number

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of edges whose removal makes the graph disconnected. A connected graph G is said to be *k-connected* if $\kappa(G) \geq k$ and *k-edge connected* if $\kappa'(G) \geq k$. The connectivity is used to measure network fault tolerance capacity.

For every integer w, $1 \leq w \leq \kappa(G)$, a *w-container* between any two distinct vertices u and v of G is a collection of 'w' internally vertex disjoint paths between them. Let $C_w(u, v)$ denote a w-container between u and v. In $C_w(u, v)$, the parameter w is the width of the container. The *length of the container* $l_w(u, v)$ is the length of a longest path among all paths in $C_w(u, v)$. The *w-wide distance* $d_w(u, v)$ between u and v is the minimum $l_w(u, v)$, over all w-containers between u and v. The *w-wide diameter* of G, $D_w(G)$ is the maximum of $d_w(u, v)$ among all pairs of vertices u, v in G, $u \neq v$. $D_{\kappa(G)}(G)$ is called the *wide diameter* of G [1]. A small wide diameter is preferred, since it enables fast multi-path communication. The wide diameter of some networks is studied in [3,4].

Fault diameter estimates the impact on the diameter of a graph, when the deletion of vertices occur. The *vertex fault diameter* denoted by $f(G)$ is defined as $f(G) = \max\{diam(G \setminus S) : S \subseteq V(G), |S| = \kappa(G) - 1\}$ [5].

Vulnerability [6,7] is a measure of the ability of the system to withstand vertex or edge faults and maximum routing delay. The maximum diameter of a graph obtained by deleting t edges from a $(t + 1)$-edge connected graph with diameter d is denoted by $f(t, d)$ and is used to study the diameter vulnerability of graphs by edge deletion.

Let G be a k-connected graph ($k \geq 1$), $\phi \neq S \subset V(G)$, and $y \in V(G \setminus S)$. A path from y to some vertex in S is called a (y, S)-path. A set of k internally disjoint (y, S)-paths is called a (y, S)-container, denoted by $C_k(G; y, S)$. The length of a longest path among all paths in $C_k(G; y, S)$ is called the length of $C_k(G; y, S)$. For a given integer $l(\geq 1)$, if there exists a (y, S)-container $C_k(G; y, S)$ with length at most l, then we say that S can (l, k)-dominate y. S is called an *(l, k)-dominating set of* G, if it (l, k)-dominates every vertex in $G \setminus S$. The set of all (l, k)-dominating sets in G is denoted by $S_{l,k}(G)$. The parameter

$$\gamma_{l,k}(G) = \min\{|S| : S \in S_{l,k}(G)\}$$

is called the (l, k)-dominating number of G and an (l, k)-dominating set S of G is called minimum if $|S| = \gamma_{l,k}(G)$ [2]. This parameter is used to characterize the reliability of resources-sharing in a network and has been recently studied in [8].

In a search for triangle-free graphs with arbitrarily large chromatic number, Mycielski developed an interesting graph transformation known as the *Mycielskian* of a graph [9]. For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with the vertex set $V(\mu(G)) = V \cup V' \cup \{w\}$, where $V' = \{u' : u \in V\}$ and the edge set $E(\mu(G)) = E \cup \{uv' : uv \in E\} \cup \{v'w : v' \in V'\}$. The vertex v' is called the twin of the vertex v and vice versa. The vertex w is called the root of $\mu(G)$. For $n \geq 2$, $\mu^n(G)$ is defined iteratively by setting $\mu^n(G) = \mu(\mu^{n-1}(G))$.

In [10], Fisher et al. studied the Hamiltonicity and diameter of the Mycielskian and proved that if G is hamiltonian, then so is $\mu(G)$ and diameter of $\mu(G) = \min(\max(2, diam(G)), 4)$. Balakrishnan and Francis Raj determined the vertex connectivity and edge connectivity of Mycielskian in [11]. Recently in [12], Guo et al. showed that for a connected graph G with $|V(G)| \geq 2$, $\mu(G)$ is super connected if and only if $\delta(G) < 2\kappa(G)$ and $\mu(G)$ is super edge connected if and only if $G \ncong K_2$. These results motivated the study of network topological properties of the Mycielskian of a graph.

In this paper, we study the wide diameter, the fault diameter, the diameter vulnerability and the (l, k)-domination of the Mycielskian of a graph. It is interesting to observe that the Mycielskian and its iterates produce large networks and preserve some nice properties of networks such as fast multi-path communication, high fault tolerance and reliable resource sharing.

The following results [11] are used in this paper.

Lemma 1.1. *For a connected graph G, $\kappa(\mu(G)) = \kappa(G) + 1$ if and only if $\delta(G) = \kappa(G)$.*

Lemma 1.2. *If G is a connected graph, then $\kappa(\mu(G)) = \kappa(G) + i + 1$ if and only if $\delta(G) = \kappa(G) + i$ for each i, $0 \leq i < \kappa(G)$.*

Lemma 1.3. *If G is a connected graph, then $\kappa(\mu(G)) = 2\kappa(G) + 1$ if and only if $\delta(G) \geq 2\kappa(G)$ and $\kappa(\mu(G)) = \min\{\delta(G) + 1, 2\kappa(G) + 1\}$.*

Lemma 1.4. *If G is a connected graph, then $\kappa'(\mu(G)) = \delta(G) + 1$.*

All graphs considered in this paper are simple, finite and undirected. For all notions not given here, see [9].