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Alpha labelings of full hexagonal caterpillars

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Abstract

Barrientos and Minion (2015) introduced the notion of generalized snake polyomino graphs and proved that when the cells are either squares or hexagons, then they admit an alpha labeling. Froncek et al. (2014) generalized the notion by introducing straight simple polyominal caterpillars with square cells and proved that they also admit an alpha labeling.

We introduce a similar family of graphs called full hexagonal caterpillars and prove that they also admit an alpha labeling. This implies that every full hexagonal caterpillar with n edges decomposes the complete graph K_{2kn+1} for any positive integer k.

Keywords: Alpha labeling; Graceful labeling; Graph decomposition

1. Introduction

At the Forty-Fifth Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton in March 2014, Minion presented her joint results with Barrientos on alpha labelings of snake polyominoes and other related graphs (they later published it in [1]). Froncek, Kingston, and Vezina [2] generalized the notion by introducing straight simple polyominal caterpillars with square cells and proved that they also admit an alpha labeling. In this paper, we introduce a similar class of graphs with hexagonal cells, called full hexagonal caterpillars and prove that they also admit an alpha labeling.

Barrientos and Minion [1] define a *snake polyomino* as a chain of m edge-amalgamated cycles $C^1, C^2, \dots, C^m$ of the same length with the property that C^1 shares one edge with C^2, C^m shares one edge with C^{m-1}, and for $i = 2, 3, \dots, m-1$, each C^i shares one edge with C^{i-1} and another edge with C^{i+1}. Note that no edge appears in more than two of those cycles. They proved that such a snake polyomino has an alpha labeling whenever the cycles are of length four or six.

Froncek, Kingston, and Vezina [2] generalized this notion for square polyominoes and defined a *straight simple polyominal caterpillar* as follows. The *spine* of the caterpillar is a straight snake polyomino in which the edges of C^i shared with C^{i-1} and C^{i+1} are non-adjacent, which means that every vertex is of degree at most three. The spine can be also viewed as the Cartesian product $P_{m+1} \square P_2$. We denote the vertices of the two paths as $x_0, x_1, \dots, x_m$

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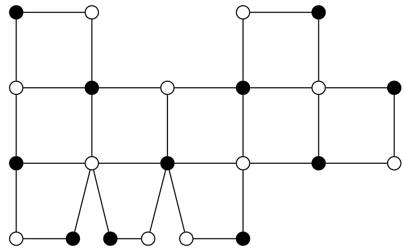

Fig. 1. Straight simple polyominal caterpillar.

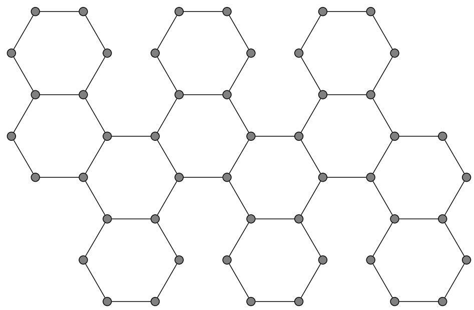

Fig. 2. Full hexagonal caterpillar H_6.

and $y_0, y_1, \ldots, y_m$, respectively. A straight simple polyominal caterpillar then can be constructed by amalgamating at most one four-cycle to each of the edges $x_j x_{j+1}$ and $y_l y_{l+1}$ for $j, l \in \{0, 1, \ldots, m-1\}$. Notice that we can amalgamate the four-cycles to none, one, or both of the two edges $x_j x_{j+1}$ and $y_j y_{j+1}$ for any admissible value of j. The number of four-cycles in the spine is the *length* of the caterpillar. An example is shown in Fig. 1.

We generalize the notion for hexagonal polyominoes in a rather restricted form and define *full hexagonal caterpillars* of length m as follows. The *spine* is a hexagonal chain (see [1]) consisting of m six-cycles, $C^1, C^2, \ldots, C^m$, where C^i consists of edges $e_1^i, e_2^i, \ldots, e_6^i$ numbered consecutively clockwise. Cycle C^1 shares edge e_5^1 with C^2, C^2 shares edge e_1^2 with C^1 and e_3^2 with C^3, C^3 shares edge e_1^3 with C^2 and e_5^3 with C^4, and so on. More precisely, we have

$$e_5^1 = e_1^2, e_3^2 = e_1^3, e_5^3 = e_1^4, \ldots, e_5^{2i-1} = e_1^{2i}, e_3^{2i} = e_1^{2i+1}, \ldots, e_5^{m-1} = e_1^m$$

when m is even and

$$e_5^1 = e_1^2, e_3^2 = e_1^3, e_5^3 = e_1^4, \ldots, e_5^{2i-1} = e_1^{2i}, e_3^{2i} = e_1^{2i+1}, \ldots, e_3^{m-1} = e_1^m$$

when m is odd.

The *legs* are six-cycles $D^1, D^2, \ldots, D^m$, where D^i consists of edges $f_1^i, f_2^i, \ldots, f_6^i$ numbered consecutively clockwise. For $i = 1, 2, \ldots, m$, cycle D^i shares edge f_1^i with C^i, where $f_1^i = e_3^i$ when i is odd, and $f_1^i = e_5^i$ when i is even. An example of a full hexagonal caterpillar H_6 of length $m = 6$ is shown in Fig. 2.

2. Supporting results and tools

Rosa [3] introduced in 1967 certain types of vertex labelings as important tools for decompositions of complete graphs K_{2n+1} into graphs with n edges.

A *labeling* ρ of a graph G with n edges is an injection from $V(G)$, the vertex set of G, into a subset S of the set $\{0, 1, 2, \ldots, 2n\}$ of elements of the additive group Z_{2n+1}. Let ρ be the injection. The *length* of an edge xy is defined as $\ell(x, y) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x)\}$. The subtraction is performed in Z_{2n+1} and hence $0 < \ell(x, y) \le n$. If the set of all lengths of the n edges is equal to $\{1, 2, \ldots, n\}$ and $S \subseteq \{0, 1, \ldots, 2n\}$, then ρ is a *rosy labeling* (called originally ρ-valuation by Rosa); if $S \subseteq \{0, 1, \ldots, n\}$ instead, then ρ is a *graceful labeling* (called β-valuation by