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Note

A note on bipartite subgraphs and triangle-independent sets

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ABSTRACT

Let $\alpha_1(G)$ denote the maximum size of an edge set that contains at most one edge from each triangle of G. Let $\tau_B(G)$ denote the minimum size of an edge set whose deletion makes G bipartite. It was first conjectured by Lehel and later independently by Puleo that $\alpha_1(G) + \tau_B(G) \leq n^2/4$ for every n-vertex graph G. Puleo showed that $\alpha_1(G) + \tau_B(G) \leq 5n^2/16$ for every n-vertex graph G. In this note, we improve the bound by showing that $\alpha_1(G) + \tau_B(G) \leq 4403n^2/15000$ for every n-vertex graph G.

1. Introduction

Let G be a simple undirected graph. A *triangle-independent set* in G is an edge set that contains at most one edge from each triangle of G. We let $\alpha_1(G)$ denote the maximum size of a triangle-independent set in G. On the other hand, a *triangle edge cover* in G is an edge set that contains at least one edge from each triangle of G. We let $\tau_1(G)$ denote the minimum size of a triangle edge cover in G.

Erdős, Gallai, and Tuza made the following conjecture:

Conjecture 1.1 (*Erdős–Gallai–Tuza [10]*). *For every n-vertex graph G, $\alpha_1(G) + \tau_1(G) \leq n^2/4$.*

Note that the equality holds for the graphs K_n and $K_{n/2,n/2}$, where n is even. Indeed, $\alpha_1(K_n) = n/2$ and $\tau_1(K_n) = \binom{n}{2} - n^2/4$ (by Mantel's theorem [13]), while $\alpha_1(K_{n/2,n/2}) = n^2/4$ and $\tau_1(K_{n/2,n/2}) = 0$. In both cases, $\alpha_1(G) + \tau_1(G) = n^2/4$. More generally, let $G_1 \vee \cdots \vee G_t$ denote the graph obtained from the disjoint union $G_1 + \cdots + G_t$ by adding all edges between vertices from different G_i. Puleo (see [16,15]) showed that the equality holds for any graph of the form $K_{r_1,r_1} \vee \cdots \vee K_{r_t,r_t}$.

Conjecture 1.1 was originally stated only for *triangular graphs*, which are graphs where every edge lies in a triangle (see [10,7]). However, later it was stated for general graphs (see [8,19]). It was proved by Puleo [15] that these two forms of the conjecture are equivalent.

A related parameter, denoted by $\tau_B(G)$, is the minimum size of an edge set in G whose deletion makes G bipartite. Clearly $\tau_B(G) \geq \tau_1(G)$. Erdős [6] asked which graphs satisfy $\tau_B(G) = \tau_1(G)$. Bondy, Shen, Thomassé, and Thomassen [3] proved that $\tau_B(G) = \tau_1(G)$ when $\delta(G) \geq 0.85\,|V(G)|$, and later Balogh, Keevash, and Sudakov [2] proved that $\tau_B(G) = \tau_1(G)$ when $\delta(G) \geq 0.79\,|V(G)|$.

The following conjecture, which is stronger than Conjecture 1.1, was first proposed by Lehel (see [7]) and later independently by Puleo [16].

Conjecture 1.2 (*[16]*). *For every n-vertex graph G, $\alpha_1(G) + \tau_B(G) \leq n^2/4$.*

Puleo [16,15] obtained many interesting results towards Conjectures 1.1 and 1.2. Conjecture 1.2 was verified for triangle-free graphs and for graphs that have no induced subgraph isomorphic to K_4^- (the graph obtained from K_4 by deleting an edge) [15]. For general graphs, Puleo [16] showed the following upper bound:

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Theorem 1.3 ([16]). *For every n-vertex graph G*, $\alpha_1(G) + \tau_B(G) \le 5n^2/16$.

The main purpose of this note is to provide an improved bound towards Conjecture 1.2. We prove that $\alpha_1(G) + \tau_B(G) \le 4403n^2/15000$ for every n-vertex graph G. We use ideas from [15,16,17], and [12]. It should be noted that after this paper was submitted, we were informed that Conjecture 1.2 was recently proved by Sergey Norin and Yue Ru Sun [14].

We shall use the following notation and terminology. For shorthand, we let $f_B(G) = \alpha_1(G) + \tau_B(G)$. We let $n(G)$, $e(G)$, and $t(G)$ denote the number of vertices, edges, and triangles in G, respectively. When there is no confusion involved, we simply write n, e, and t. We let $d(v)$ denote the degree of a vertex v, and $\omega(G)$ denote the clique number of G. When $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S, $\overline{S}$ for the set $V(G) - S$, and $[S, \overline{S}]$ for the set of all edges with one endpoint in S and the other endpoint in $\overline{S}$. We use the term *minimal counterexample* to refer to a vertex-minimal counterexample, that is, a graph G such that the property in question holds for every proper induced subgraph of G but does not hold for G.

The rest of the paper is organized as follows. In the next section, we investigate the structure of a minimal counterexample to $f_B(G) \le cn(G)^2$ where $c > 1/4$. We show that the clique number of such a counterexample is bounded by a function of c. Thus, to prove that $f_B(G) \le cn(G)^2$, we only need to prove it for graphs with small clique number. Then in Section 3 we present a quick proof of $f_B(G) \le 3n(G)^2/10$, which improves Theorem 1.3. In Section 4 we give some bounds on $\tau_B(G)$ for K_6-free graphs. In particular, we show that every n-vertex K_6-free graph can be made bipartite by deleting at most $17n^2/100$ edges. In Section 5 we prove our main result.

2. $f_B(G)$ and clique number

We need the following lemma from [15].

Lemma 2.1 ([15]). *Let G be a graph, and let A be a triangle-independent set of edges in G. If S is a nonempty proper subset of V(G), then*

$$f_B(G) \le f_B(G[S]) + f_B(G[\overline{S}]) + \frac{1}{2}\left|[S, \overline{S}]\right| + \left|[S, \overline{S}] \cap A\right|.$$

In [15], Puleo used Lemma 2.1 to prove some conclusions on the structure of a minimal counterexample G to Conjecture 1.2. By slightly extending his argument, we show the following:

Lemma 2.2. *For any constant $c > 1/4$, if G is a minimal counterexample to $f_B(G) \le cn(G)^2$, then $\omega(G) < 1/(4c-1)$.*

Proof. Let G be a minimal counterexample to $f_B(G) \le cn(G)^2$. We may assume $n(G) \ge 5$, since it is easy to verify that $f_B(G) \le n(G)^2/4 \le cn(G)^2$ when $n(G) \le 4$. Let K be the largest clique in G, and let $k = |K| = \omega(G)$. Since $f_B(G) \le n(G)^2/4 \le cn(G)^2$ when G is complete, we may assume $1 \le k \le n(G) - 1$.

For simplicity, write n for $n(G)$. Let A be any triangle-independent set in G, and for every $v \in V(G)$, let $N_A(v) = \{w \in V(G) : vw \in A\}$. Since A is triangle-independent, $|N_A(v) \cap K| \le 1$ for each $v \in \overline{K}$. It follows that $\left|[K, \overline{K}] \cap A\right| \le n - k$. By Lemma 2.1 and the minimality of G, we have

$$\begin{aligned} cn^2 < f_B(G) &\le f_B(G[K]) + f_B(G[\overline{K}]) + \frac{1}{2}\left|[K, \overline{K}]\right| + \left|[K, \overline{K}] \cap A\right| \\ &\le \frac{k^2}{4} + c(n-k)^2 + \frac{1}{2}\left|[K, \overline{K}]\right| + n - k. \end{aligned}$$

Thus, $\left|[K, \overline{K}]\right| > -(2c + \frac{1}{2})k^2 + 4cnk + 2k - 2n$. However, since K is the largest clique of G, $\left|[K, \overline{K}]\right| \le (n-k)(k-1)$. Hence, we have

$$(n-k)(k-1) > -\left(2c + \frac{1}{2}\right)k^2 + 4cnk + 2k - 2n.$$

The inequality above simplifies to $(\frac{1}{2} - 2c)k^2 + k < (1 - (4c-1)k)\,n$. Suppose to the contrary that $k \ge 1/(4c-1)$. Then $(1 - (4c-1)k)\,n \le (1 - (4c-1)k)\,k$. It follows that $(\frac{1}{2} - 2c)k^2 + k < (1 - (4c-1)k)\,k$. That is, $c < 1/4$, a contradiction. □

3. A first improvement

In this section we present a quick proof of $f_B(G) \le 3n(G)^2/10$. We first show that the conclusion holds for K_5-free graphs, and then use Lemma 2.2 to prove that it holds for all graphs.

For a graph G, let $b(G)$ denote the largest size of a vertex set B such that B induces a bipartite subgraph of G. Puleo [16] proved the following bound on $\alpha_1(G)$:

Lemma 3.1 ([16]). *For every n-vertex graph G*, $\alpha_1(G) \le nb(G)/4$.