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On ve-degrees and ev-degrees in graphs[☆]

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ABSTRACT

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A vertex $v \in V$ ve-dominates every edge incident to it as well as every edge adjacent to these incident edges. The vertex–edge degree of a vertex v is the number of edges ve-dominated by v. Similarly, an edge $e = uv$ ev-dominates the two vertices u and v incident to it, as well as every vertex adjacent to u or v. The edge–vertex degree of an edge e is the number of vertices ev-dominated by edge e. In this paper we introduce these types of degrees and study their properties.
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1. Introduction

In this paper we study some aspects of the vertex–edge degree of a vertex and the edge–vertex degree of an edge, particularly with regard to the vertex–edge and edge–vertex counterparts of graph regularity and irregularity.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Given $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. A vertex v *ve-dominates* every edge uv incident to v, as well as every edge adjacent to these incident edges, that is, each edge incident to a vertex in $N[v]$. Similarly, an edge $e = uv$ *ev-dominates* the vertices incident to it (namely, u and v) as well as the vertices adjacent to u or v, that is, each vertex in $N[u] \cup N[v] = N(u) \cup N(v)$. There is a natural duality between ve-domination and ev-domination: in any graph G, a vertex $v \in V$ ve-dominates an edge $e \in E$ if and only if the edge e ev-dominates vertex v.

A set $S \subseteq V$ is a *vertex–edge dominating set* (or simply, a *ve-dominating set*) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v ve-dominates e. Similarly, a set $M \subseteq E$ is an *edge–vertex dominating set* (or simply, an *ev-dominating set*) if for every vertex $v \in V$, there exists an edge $e \in E$ such that e ev-dominates v. The concepts of vertex–edge domination and edge–vertex domination were introduced by Peters [14] in his 1986 Ph.D. thesis and studied further in [4,9,10].

☆ We dedicate this paper to our good friend Gary Chartrand for his seminal questions on irregularity.

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The *ve-degree* of a vertex $v \in V$, denoted $\deg_{ve}(v)$, equals the number of edges ve-dominated by v. The *ev-degree* of an edge $e = uv$, denoted $\deg_{ev}(e)$, equals the number of vertices ev-dominated by e. When necessary we will use the notation $\deg_G^{ve}(v)$ and $\deg_G^{ev}(e)$. To help fix these definitions consider the *house graph*, consisting of a five cycle, with vertices a–b–c–d–e–a, and the chord eb. For this graph, $\deg_{ve}(a) = \deg_{ve}(c) = \deg_{ve}(d) = 5$, and $\deg_{ve}(b) = \deg_{ve}(e) = 6$. Similarly, for the six edges, $\deg_{ev}(ab) = \deg_{ev}(cd) = \deg_{ev}(ea) = 4$, and $\deg_{ev}(bc) = \deg_{ev}(de) = \deg_{ev}(eb) = 5$. Notice that the sum of the ve-degrees is 27 and the sum of the ev-degrees is 27 as well. This is an example of a general phenomenon that we will address in Section 2.

A graph with two or more vertices is called *regular* if its vertices have the same degree and *irregular* if no two vertices have the same degree. A graph G is *ve-regular* if all its vertices have the same ve-degree and *ev-regular* if all its edges have the same ev-degree. A graph G is called *ve-irregular* if no two vertices in V have the same ve-degree, that is, $\deg_{ve}(u) \neq \deg_{ve}(v)$ for all $u, v \in V, u \neq v$. A graph G is called *ev-irregular* if no two edges in E have the same ev-degree, that is, $\deg_{ev}(e) \neq \deg_{ev}(f)$ for all $e, f \in E, e \neq f$.

The remainder of this paper is organized as follows. In Section 2 we investigate properties of ve-degrees and ev-degrees. Among other things we derive the analog of the degree-sum formula and show that there exist graphs having an odd number of vertices of odd ve-degree as well as graphs having an odd number of edges of odd ev-degree. In Sections 3 and 4 we study graph regularity and irregularity for these types of degrees. In particular we characterize the regular graphs for some low ve and ev-degrees. Furthermore, while Behzad and Chartrand [3] have shown that no graph is irregular, we show that no connected graph is ev-irregular but that ve-irregular graphs do exist.

2. Properties of *ve*-degrees and *ev*-degrees

Simple upper and lower bounds for the ve-degrees and the ev-degrees of a graph can be specified in terms of its size and order. Let G be a connected graph of order n, $n \geq 3$, and size m. Then for any vertex $v \in V$ and edge $e = xy \in E$, $2 \leq \deg_{ve}(v) \leq m$ and $3 \leq \deg_{ev}(e) \leq n$. Each of the extreme values corresponds to a local or global structure in the graph G: $deg_{ve}(v) = 2$ if and only if either v is the center of G and G is the path P_3 or v is a leaf whose support vertex has degree two; $\deg_{ve}(v) = m$ if and only if v ve-dominates G; $\deg_{ev}(e) = 3$ if and only if either $\deg(x) = \deg(y) = 2$ and x and y are contained in a triangle, or one of x or y is a leaf and the other is its support vertex of degree two; finally, $\deg_{ev}(e) = n$ if and only if e ev-dominates G.

Our first result contains the vertex–edge and edge–vertex analogs of the degree sum formula, which says that in any graph the sum of the degrees of the vertices is equal to twice the number of edges. Given $e \in E$, let η_e denote the number of triangles in G containing the edge e and let $\eta(G)$ denote the total number of triangles in G.

Theorem 1. *For any graph G,*

$$\sum_{v\in V} \deg_{ve}(v) = \sum_{e\in E} \deg_{ev}(e) = \left(\sum_{v\in V} \deg^2(v)\right) - 3\eta(G).$$

Proof. The first equality is straightforward: a vertex $v \in V$ ve-dominates an edge $e \in E$ if and only if e ev-dominates v; hence, $\sum_{v\in V} \deg_{ve}(v) = \sum_{e\in E} \deg_{ev}(e)$, since both sides count the pairs (v, e) where v ve-dominates e.

Concerning the sum of the squares of the degrees, let $e = uv \in E$ and note that $\deg_{ev}(e) = |N(u)| + |N(v)| - |N(u) \cap N(v)|$. Recalling that η_e denotes the number of triangles in G containing the edge e, we can rewrite this as

$$\deg_{ev}(e) = \deg(u) + \deg(v) - \eta_e.$$

Thus $\sum_{e\in E} \deg_{ev}(e) = \sum_{e\in E}(\deg(u) + \deg(v)) - \sum_{e\in E} \eta_e$.

To finish our proof first observe that $\sum_{e\in E} \eta_e = 3\eta(G)$, since each triangle will be counted three times, once for each of its edges. Finally, since each vertex $w \in V$ is incident to $\deg(w)$ edges in G, the term $\deg(w)$ will appear $\deg(w)$ times in the sum $\sum_{e\in E}(\deg(u) + \deg(v))$; consequently,

$$\sum_{e\in E}(\deg(u) + \deg(v)) = \sum_{v\in V} \deg^2(v). \tag{1}$$

In summary

$$\sum_{e\in E} \deg_{ev}(e) = \left(\sum_{v\in V} \deg^2(v)\right) - 3\eta(G),$$

as was to be shown. □

In 1972 Gutman and Trinajstić implicitly identified two topological indices (graphical invariants) to study the dependence of the Hückel total π-electron energy on molecular structure [7]. These indices, customarily called the Zagreb indices, were formally introduced in 1975 in a paper with Wilcox [6]. The Zagreb indices are easily defined: given a graph $G = (V, E)$, let

$$M_1 = \sum_{v\in V} \deg^2(v) \quad \text{and} \quad M_2 = \sum_{uv\in E} \deg(u)\deg(v).$$

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