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Note

Asymptotically optimal neighbor sum distinguishing total colorings of graphs

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ABSTRACT

Given a proper total k-coloring $c : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ of a graph G, we define the *value* of a vertex v to be $c(v) + \sum_{uv \in E(G)} c(uv)$. The smallest integer k such that G has a proper total k-coloring whose values form a proper coloring is the *neighbor sum distinguishing total chromatic number* of G, $\chi''_{\Sigma}(G)$. Pilśniak and Woźniak (2013) conjectured that $\chi''_{\Sigma}(G) \le \Delta(G) + 3$ for any simple graph with maximum degree $\Delta(G)$. In this paper, we prove this bound to be asymptotically correct by showing that $\chi''_{\Sigma}(G) \le \Delta(G)(1 + o(1))$. The main idea of our argument relies on Przybyło's proof (2014) regarding neighbor sum distinguishing edge-colorings.

1. Introduction

A *proper k-coloring* of a graph G is a function $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$, where $[k]$ denotes $\{1, \dots, k\}$. The least k such that G has a proper k-coloring is the *chromatic number* of G, $\chi(G)$. Clearly $\chi(G) \le \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. A *proper k-edge-coloring* of G is a function $c : E(G) \to [k]$ such that $c(uv) \neq c(uw)$ when $v \neq w$. The least k such that G has a proper k-edge-coloring is the *edge-chromatic number* of G, $\chi'(G)$. For a k-edge-coloring c, define the *value* $s_c(v)$ of a vertex v by $\sum_{u \in N(v)} c(uv)$. A k-edge-coloring c of G is a *proper edge-weighting* if s_c forms a proper coloring of G. The least k such that G has a proper k-edge-coloring that is a proper edge-weighting is the *neighbor sum distinguishing edge-chromatic number* of a graph, denoted $\chi'_{\Sigma}(G)$. Note that this graph parameter is well defined only for graphs with no isolated edges.

The study of this type of graph invariants, focused on distinguishing vertices by their associated sums of colors of e.g. their incident edges was initiated in [4], where the parameter called the *irregularity strength* of a graph was introduced. Within this, motivated by the trivial fact that no irregular graph exists (with 1-vertex exception) and with reference to research in [3] of Chartrand, Erdős and Oellermann on possible alternative definitions of such a graph, Chartrand et al. tried to propose a measure of irregularity of a given graph G by means of multiplying edges of G in order to produce an irregular multigraph of G, see [4] for details. The same construction was also expressed in terms of colorings, where integer colors of edges of a graph corresponded to multiplicities of the edges in a corresponding multigraph, while the sums of these colors from the edges incident with a given vertex – to the degree of this vertex in the multigraph. The irregularity strength is a well studied graph invariant, but it also give rise to a whole discipline, which might be referred to as additive graph labelings or more generally – vertex distinguishing graph colorings, see e.g. [7], including extensive study of these.

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For the edge-coloring parameters, $\chi'_\Sigma(G) \geq \chi'(G) \geq \Delta(G)$, while by Vizing's Theorem, $\chi'(G) \leq \Delta(G) + 1$ for every graph G. Flandrin, Marczyk, Przybyło, Saclé, and Woźniak [6] conjectured that:

Conjecture 1.1 ([6]). *If G is a connected graph with at least three vertices other than C_5, then $\chi'_\Sigma(G) \leq \Delta(G) + 2$.*

Przybyło [14] proved an asymptotically optimal upper bound for graphs with large maximum degree. Specifically, he showed:

Theorem 1.2 ([14]). *If G is a connected graph with $\Delta(G)$ sufficiently large, then $\chi'_\Sigma(G) \leq \Delta(G) + 50\Delta(G)^{5/6}\ln^{1/6}\Delta(G)$.*

A *proper total k-coloring* of G is a function $c : V(G) \cup E(G) \to [k]$ such that c restricted to $V(G)$ is a proper coloring, c restricted to $E(G)$ is a proper edge-coloring, and such that the color on each vertex is different from the color on its incident edges. The least number of colors in such a coloring of G is denoted by $\chi''(G)$. For a proper total k-coloring c, define the *value* $s_c(v)$ of a vertex v by $c(v) + \sum_{u \in N(v)} c(uv)$. A proper total k-coloring c of G is a *proper total weighting* if s_c is a proper coloring of G. The least k such that G has a proper total k-coloring that is a proper total weighting is the *neighbor sum distinguishing total chromatic number* of G, denoted $\chi''_\Sigma(G)$. Clearly, $\chi''_\Sigma(G) \geq \chi''(G) \geq \Delta(G) + 1$. On the other hand, the famous Total Coloring Conjecture, that has eluded mathematicians for half a century presumes that $\chi''(G) \leq \Delta(G) + 2$ for every graph G. This was independently posed by Vizing [15] and Behzad [2]. Thus far, it has been confirmed up to a (large) additive constant by means of the probabilistic method, see [10]. Despite that, Pilśniak and Woźniak [13] daringly conjectured the following.

Conjecture 1.3 ([13]). *If G is a graph with maximum degree $\Delta(G)$, then $\chi''_\Sigma(G) \leq \Delta(G) + 3$.*

Pilśniak and Woźniak [13] proved that Conjecture 1.3 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. Using the Combinatorial Nullstellensatz, Wang, Ma, and Han [16] proved that the conjecture holds for triangle-free planar graphs with maximum degree at least 7. Dong and Wang [5] showed that Conjecture 1.3 holds for sparse graphs, and Li, Liu, and Wang [9] proved that the conjecture holds for K_4-minor free graphs. Li, Ding, Liu, and Wang [8] also confirmed Conjecture 1.3 for planar graphs with maximum degree at least 13. Finally, Xu, Wu, and Xu [17] proved $\chi''_\Sigma(G) \leq \Delta(G) + 2$ for graphs G with $\Delta(G) \geq 14$ that can be embedded in a surface of nonnegative Euler characteristic.

By modifying Przybyło's proof that Conjecture 1.1 is asymptotically correct for graphs with large maximum degree, we confirm in this paper that Conjecture 1.3 is also asymptotically correct by showing the following.

Theorem 1.4. *If G is a graph with $\Delta(G)$ sufficiently large, then $\chi''_\Sigma(G) \leq \Delta(G) + 50\Delta(G)^{5/6}\ln^{1/6}\Delta(G)$.*

2. Ideas

We color the vertices of the graph first and produce an edge-coloring such that the combined total coloring is a proper total weighting. For a coloring g and an edge-coloring h, let $g \sqcup h$ be the total coloring produced by combining g and h.

Our main work is in producing the desired edge-coloring. Our Lemma 2.3 serves a similar purpose to Lemma 6 of Przybyło [14]. Lemma 2.3 guarantees (not necessarily proper) colorings c_1 and c_2 of the vertices and edges respectively such that the colors are roughly evenly distributed. These colorings are used to produce an initial (also improper) edge-coloring c' by setting $c'(uv) = c_1(u) + c_1(v) + c_2(uv)$. Statement (3) of Lemma 2.3 guarantees that the colors used by c' are also roughly evenly distributed. Finally, statement (4) of Lemma 2.3 will be used to guarantee that the final values for the vertices form a proper coloring. The proof uses the Lovász Local Lemma and the Chernoff Bound in the forms below.

Theorem 2.1 (*Lovász Local Lemma [1]*). *Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all but at most D others of these events, and that $\mathbb{P}(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(D+1) \leq 1$, then $\mathbb{P}(\bigcap_{i=1}^n \overline{A_i}) > 0$.*

Theorem 2.2 (*Chernoff Bound [12]*). *If $0 \leq t \leq np$, then*

$$\mathbb{P}(|BIN(n,p) - np| > t) < 2e^{-t^2/3np},$$

where $BIN(n,p)$ is a binomial random variable with n independent trials having success probability p.

For a vertex v, let $d(v)$ be the degree of v.

Lemma 2.3. *Let H be a graph of maximum degree at most D. For any coloring c_1 of $V(H)$ and a vertex $v \in V(H)$ with $d = d(v)$, let*

$$S(v) = \left(\lceil D^{2/3}\ln^{1/3} D\rceil + \lceil D^{1/2}\rceil\right) dc_1(v) + R(d, D),$$

where R is a function of two variables.

If D is sufficiently large, then there exist colorings $c_1 : V(H) \to \left[\lceil D^{1/6}\ln^{-1/6} D\rceil\right]$ and $c_2 : E(H) \to \left[\lceil D^{1/3}\ln^{-1/3} D\rceil\right]$ such that if $S(v) \leq D^2$ for all $v \in V(H)$, then for every vertex v: