Discrete Mathematics 340 (2017) 63–71

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Arc-transitive antipodal distance-regular covers of complete graphs related to $SU_3(q)$

L.Yu. Tsiovkina

Krasovsky Institute of Mathematics and Mechanics of UB RAS, Yekaterinburg, 620990, Russia

ARTICLE INFO

Article history:
Received 2 August 2015
Received in revised form 28 July 2016
Accepted 1 August 2016

Keywords:
Antipodal graph
Arc-transitive graph
Automorphism group
Distance-regular graph
Covering graph

ABSTRACT

In this paper, we classify antipodal distance-regular graphs of diameter three that admit an arc-transitive action of $SU_3(q)$. In particular, we find a new infinite family of distance-regular antipodal r-covers of a complete graph on $q^3 + 1$ vertices, where q is odd and r is any divisor of $q + 1$ such that $(q + 1)/r$ is odd. Further, we find several new constructions of arc-transitive antipodal distance-regular graphs of diameter three in case $\lambda = \mu$.

1. Introduction

Antipodal distance-regular covers of complete graphs form a vast subclass of imprimitive distance-regular graphs. Such covers of diameter three are of a highly complicated structure, which in general cannot be regained by their antipodal quotients. A promising approach on the way of their classification consists in a description of those covers, which possess some specific or "rich" groups of automorphisms. For example, antipodal distance-regular graphs of diameter three with the covering group acting regularly on an antipodal class were characterized in terms of matrices over group algebras by Godsil and Hensel [8], and later were given a description through generalized Hadamard matrices by Klin and Pech [10] in a special case, which led to the discovery of new examples of such graphs. Furthermore, distance-transitive antipodal covers of complete graphs are all known (up to isomorphism) due to result by Godsil, Liebler and Praeger [9].

This paper concerns the problem of classification of arc-transitive antipodal distance-regular covers of complete graphs (these covers, clearly, include distance-transitive ones). Symmetry properties of such a graph together imply that its automorphism group G induces a 2-transitive permutation group G^{Σ} on its antipodal classes (see [9, Lemma 2.6]). The classification of the finite 2-transitive permutation groups plays a crucial role when solving this problem, by dividing it naturally up into two cases: G^{Σ} is an almost simple group or an affine group. The study of covers in the affine case was almost completed by Makhnev, Paduchikh, and the present author [13,22]. The case of covers with almost simple group G^{Σ} was settled by the same authors in [12,21] under the condition $\lambda = \mu$ and with an exception (which is treated in the present work) in case $PSU_3(q) \leq G$.

In this paper, we completely describe antipodal distance-regular graphs of diameter three that admit an arc-transitive group of automorphisms, isomorphic to $SU_3(q)$ or $PSU_3(q)$ (see Theorem 3.9). This is motivated by the fact that, as it is shown by further study, the classification of antipodal distance-regular graphs of diameter three with almost simple group G^{Σ} in

E-mail address: l.tsiovkina@gmail.com.

http://dx.doi.org/10.1016/j.disc.2016.08.001

a sense can be reduced to the case $soc(G^{\Sigma}) \simeq PSU_3(q)$. In view of results of Taylor [19], Cameron [4], Godsil et al. [9], and Brouwer [3], it appears that the 3-dimensional special unitary groups are the most productive source of such graphs.

To establish distance-regularity of certain graphs, we elaborate a technique, similar to that of [21]. With the use of it, we find a new infinite family of distance-regular graphs with $\lambda \neq \mu$ (see Theorem 3.9). Their constructions are considered in Section 3. These graphs also can be regarded as distance-regular covers of some Cameron graphs. It turns out that, in general, taking into account the results of [21], a canonical form of elements in 2-transitive finite simple groups to a high extent determines the structure of associated arc-transitive covers of complete graphs, which allows, for example, one to study their matchings and local graphs.

Another purpose of this paper is to find new constructions of several families of arc-transitive antipodal distance-regular covers of complete graphs with $\lambda = \mu$, which, in turn, reflect an internal geometry of related 2-transitive groups (see Section 4). By generalizing some previously known constructions, we obtain that the graph, whose vertices are the involutions of $G \in \{Sz(q), PSU_3(q), PSL_2(q)\}$, where $q > 2$ is even, and whose edges are the pairs $\{u, v\}$ such that uv has order $\chi(G)$, where $\chi(G)$ denotes the associated prime number in the sense of Suzuki, is distance-regular (see Proposition 4.1). Then, some other constructions of distance-regular graphs on a conjugacy class of non-trivial elements from the centres of the Sylow p-subgroups of $G \in \{^2G_2(q), PSU_3(q)\}$, where $q > 3$ is odd, are presented (see Proposition 4.5).

The paper is organized as follows. Section 2 contains some necessary definitions and results on arc-transitive graphs, coverings and the structure of groups $PSL_2(q)$ (for even q) and $PSU_3(q)$. In Section 3 we prove Theorem 3.9. Finally, in Section 4 we prove Propositions 4.1 and 4.5.

2. Terminology and preliminaries

Our group theoretic terminology (except for some specific notions, which are explained shortly after their introduction) and notation are mostly standard and follow [1] and [6]. In this paper, a graph is understood to be undirected graph without loops or multiple edges, unless otherwise specified. Let Γ be a graph. Then by $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ we denote the vertex set and the automorphism group of Γ, respectively. For a vertex $x \in V(\Gamma)$ by $\Gamma_i(x)$ we denote the set of vertices of Γ at distance i from x, and if Γ is fixed, then we will write $[x]$ instead of $\Gamma_1(x)$.

Let Γ be a connected graph of diameter d and $x \in V(\Gamma)$. Take $y \in \Gamma_i(x)$ and set $b_i(x, y) = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$, $a_i(x, y) = |\Gamma_i(x) \cap \Gamma_1(y)|$ and $c_i(x, y) = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ (we assume that $b_d(x, y) = c_0(x, y) = 0$). The graph Γ is said to be distance-regular if, for all $0 \leq i \leq d$, each of the parameters $b_i(x, y)$, $a_i(x, y)$ and $c_i(x, y)$ depends not on the choice of x and y but only on i. In that case, the sequence $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, where $b_i = b_i(x, y)$ and $c_i = c_i(x, y)$, is called the intersection array of Γ. We will denote parameters a_1 and c_2 by λ and μ, respectively.

A connected graph Γ of diameter $d > 1$ is said to be antipodal, if the (antipodality) relation "to be at distance 0 or d" is an equivalence relation on $V(\Gamma)$, and the classes of this relation are called antipodal classes.

Now let Γ be an arbitrary graph. Suppose there is a partition $\Sigma = \{C_1, C_2, \dots, C_n\}$ of $V(\Gamma)$ such that each C_i induces a coclique of Γ, and for each pair $\{C_i, C_j\}$ such that $i \neq j$ either there are no edges between C_i and C_j, or there is a matching. Then one can consider the graph $\Gamma/_{\Sigma}$ with the vertex set Σ and with two vertices C_i and C_j being adjacent if and only if there is a matching between C_i and C_j in Γ. Then Γ is called a covering graph of $\Gamma/_{\Sigma}$ and the elements of Σ are called fibres (cf. [8]), and if all fibres of Σ have the same size r, then Γ is called r-fold covering graph of $\Gamma/_{\Sigma}$. If Γ and Σ are fixed, then for each $x \in V(\Gamma)$ we will denote a fibre F from Σ by $F(x)$ whenever $x \in F$. In addition, if Γ is antipodal and Σ coincides with the partition of $V(\Gamma)$, defined by the antipodality relation on $V(\Gamma)$, and if r denotes the size of a fibre in Σ, then Γ is called an antipodal r-cover of $\Gamma/_{\Sigma}$. If Γ is an antipodal distance-regular graph of diameter three, then Γ is an antipodal r-cover of a complete graph on n vertices, and Γ has intersection array $\{n-1, \mu(r-1), 1; 1, \mu, n-1\}$ (for the further theory of such covers, see [8,10] and [2]).

A graph is called arc-transitive if its automorphism group acts transitively on ordered pairs of adjacent vertices.

From now and on, for a natural number n and a prime α by $(n)_\alpha$ we denote the maximal power of α dividing n. Let G be a group. Then for all elements $g, x \in G$ by g^x and g^{-x} we denote the elements $x^{-1}gx$, or $(g^{-1})^x$, respectively. If A and B are some non-empty subsets of G, then we put $A^B = \{g^x | g \in A, x \in B\}$. We also write $G^{(1)}$ to denote the commutator group of G and put $G^{\#} = G - \{1\}$.

The next proposition contains some important basic facts on arc-transitive graphs and covers.

Proposition 2.1 ([9]). *Suppose that a non-normal subgroup H of a group G and an element $g \in G - H$ are given. Let $\Gamma(G, H, HgH)$ denote the graph (possibly, directed) with vertex set $R(G, H) = \{Hx : x \in G\}$ whose edges are the pairs $\{Hx, Hy\}$ such that $xy^{-1} \in HgH$.*

(1) *If G acts faithfully on $R(G, H)$, $g^2 \in H$ and $G = \langle H, g\rangle$, then $\Gamma(G, H, HgH)$ is an (undirected) connected graph that admits the group G acting (by right multiplication) faithfully and transitively both on vertices and on arcs.*
(2) *Suppose that G acts arc-transitively on a connected graph Γ, H is the stabilizer of a vertex x of Γ in G and g is a 2-element such that x and x^g are adjacent. Then $\Gamma \simeq \Gamma(G, H, HgH)$, $g^2 \in H$ and $G = \langle H, g\rangle$.*