Discrete Mathematics 340 (2017) 72–80

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Monochromatic cycle power partitions☆

Gábor N. Sárközy*

Computer Science Department, Worcester Polytechnic Institute, Worcester, MA, 01609, USA
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, P.O. Box 127, Budapest, H-1364, Hungary

ARTICLE INFO

Article history:
Received 7 February 2016
Received in revised form 3 August 2016
Accepted 4 August 2016

Keywords:
Monochromatic partitions
Powers of cycles

ABSTRACT

Improving our earlier result we show that for every integer $k \geq 1$ there exists a $c(k)$ such that in every 2-colored complete graph apart from at most $c(k)$ vertices the vertex set can be covered by $200k^2 \log k$ vertex disjoint monochromatic kth powers of cycles.

1. Monochromatic partitions and powers of cycles

K_n is the complete graph on n vertices and $K_{n,n}$ is the complete bipartite graph between two sets of n vertices each. If $G_1, G_2, \ldots, G_r$ are graphs, then the **Ramsey number** $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one of the subgraphs H_i $(1 \leq i \leq r)$ has a subgraph isomorphic to G_i. In this paper we will deal with 2-color Ramsey numbers (so $r = 2$) and we will think of color 1 as red and color 2 as blue. The kth **power** of a cycle of length n, denoted C_n^k, is the graph obtained from C_n by joining every pair of vertices with distance at most k (counting edges) in C. For simplicity let us call the kth power of a cycle a k-cycle.

Assume first that K_n is a complete graph on n vertices whose edges are colored with r colors $(r \geq 1)$. How many monochromatic cycles are needed to partition the vertex set of K_n? This question received a lot of attention in the last few years. Throughout the paper, single vertices and edges are considered to be cycles. Let $p(r)$ denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored K_n. It is not obvious that $p(r)$ is a well-defined function. That is, it is not obvious that there always is a partition whose cardinality is independent of the order of the complete graph. However, in [8] Erdős, Gyárfás and Pyber proved that there exists a constant c such that $p(r) \leq cr^2 \log r$ (throughout this paper log denotes natural logarithm). Furthermore, in [8] (see also [15]) the authors conjectured the following.

Conjecture 1. $p(r) = r$.

The special case $r = 2$ of this conjecture was asked earlier by Lehel and for $n \geq n_0$ was first proved by Łuczak, Rödl and Szemerédi [28]. Allen improved on the value of n_0 [1] and finally Bessy and Thomassé [4] proved the original conjecture for $r = 2$. For general r the current best bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [17] who proved that for $n \geq n_0(r)$ we have $p(r) \leq 100r \log r$. For $r = 3$ an approximate version of the conjecture was proved in [18] but surprisingly Pokrovskiy [29] found a counterexample to that conjecture. However, in the counterexample all but one vertex can be covered by r vertex disjoint monochromatic cycles. Thus a slightly weaker version of the conjecture may still be true,

☆ The author was supported in part by OTKA Grant K104373.
* Correspondence to: Computer Science Department, Worcester Polytechnic Institute, Worcester, MA, 01609, USA.
E-mail address: gsarkozy@cs.wpi.edu.

http://dx.doi.org/10.1016/j.disc.2016.08.009

namely that apart from a constant number of vertices the vertex set can be covered by r vertex disjoint monochromatic cycles.

Conjecture 2. *Let G be a r-colored graph. Then there exist a constant $c = c(r)$ and r vertex disjoint monochromatic cycles of G that cover at least $n - c$ vertices.*

Let us also note that the above problem was generalized in various directions; for hypergraphs (see [19] and [31]), for complete bipartite graphs (see [8] and [20]), for graphs which are not necessarily complete (see [3] and [30]) and for vertex partitions by monochromatic connected k-regular subgraphs (see [32] and [33]).

Another area that attracted a lot of interest is powers of cycles; in particular the famous Pósa–Seymour conjecture.

Conjecture 3. *If the minimum degree of a graph G on n vertices is at least $\frac{k}{k+1}n$, then G contains the kth power of a Hamiltonian cycle, i.e. a Hamiltonian k-cycle.*

After a sequence of partial results [9–13,24] with an application of the Regularity Lemma-Blow-up Lemma method we showed in [21] and [25] that the conjecture is true for graphs with $n \geq n_0$. Since we used the Regularity Lemma, the resulting n_0 was huge. Later for $k = 2$ in [27] we "deregularized" the proof, i.e. we eliminated the use of the Regularity Lemma from the proof and thus the resulting n_0 was much better. This was further improved in [7].

In [2] Allen, Brightwell and Skokan studied the Ramsey number $R(C_n^k, C_n^k)$ where again C_n^k is a k-cycle on n vertices. They proved the following lower bounds

$$R(C^k_{(k+1)t}, C^k_{(k+1)t}) \geq t(k+1)^2 - 2k \quad \text{for } k \geq 2,$$

and

$$R(C^k_{(k+1)t+r}, C^k_{(k+1)t+r}) \geq (k+1)((k+2)t + 2r - 2) + r \quad \text{for } k \geq 2, 1 \leq r \leq k,$$

and they conjectured that these bounds are, at least asymptotically, optimal. However, they were able to prove only the following upper bound

$$R(C_n^k, C_n^k) \leq \left(2\chi(C_n^k) + \frac{2}{\chi(C_n^k)}\right) n + o(n), \tag{1}$$

(where $\chi(G)$ denotes the chromatic number of graph G) which differs from the lower bounds by a multiplicative factor slightly greater than 2.

A natural question (first asked by András Gyárfás) is to combine the above two areas and ask how many monochromatic k-cycles are needed to partition the vertex set of a 2-colored K_n. In an earlier paper [14] (as a consequence of a more general theorem) we showed that $2^{ck \log k}$ monochromatic k-cycles are enough. Here we improve this significantly to $ck^2 \log k$ but the price we have to pay is that a constant number of vertices might be left uncovered (similarly to Conjecture 2).

Theorem 1. *For every integer $k \geq 1$ there exists a $c(k)$ such that in every 2-colored complete graph apart from at most $c(k)$ vertices the vertex set can be covered by $200k^2 \log k$ vertex disjoint monochromatic k-cycles.*

Unfortunately, the number $c(k)$ of uncovered vertices is quite large in terms of k; it is a Regularity Lemma-type quantity. If possible, it would be desirable to eliminate these uncovered vertices. Furthermore, we believe that in light of (1) (since the Ramsey number is $O(kn)$) the right number of k-cycles is probably linear in k. Finally, it would be interesting to extend this problem for more than 2 colors.

2. Notation and tools

For basic graph concepts see the monograph of Bollobás [5].

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A \cup B$, and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $N(v)$ is the set of neighbors of $v \in V$. Hence $|N(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. For $A \subset V(G)$ we write $N(A) = \cap_{v \in A} N(v)$, the set of common neighbors. $N(x, y, z, \ldots)$ is shorthand for $N(\{x, y, z, \ldots\})$. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. For non-empty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the **density** of the graph between A and B. The density of the graph G is $d(G) = |E(G)|/\binom{n}{2}$. We say that the graph G is δ-**dense** if $d(G) \geq \delta$.

Definition 1. The bipartite graph $G = (A, B, E)$ is ε-**regular** if

$$X \subset A, \ Y \subset B, \ |X| > \varepsilon|A|, \ |Y| > \varepsilon|B| \ \text{ imply } \||d(X, Y) - d(A, B)| < \varepsilon,$$

otherwise it is ε-irregular.