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A negative answer to a problem on generalized Fibonacci cubes[☆]

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ABSTRACT

Generalized Fibonacci cube $Q_n(f)$ is the graph obtained from the n-cube Q_n by removing all vertices that contain a given binary string f as a consecutive substring. A binary string f is called bad if $Q_n(f)$ is not an isometric subgraph of Q_n for some n, and the smallest such integer n, denoted by $B(f)$, is called the index of f. Ilić, Klavžar and Rho posed a problem that if $Q_n(f)$ is not an isometric subgraph of Q_n, is there a dimension n' such that $Q_n(f)$ can be isometrically embedded into $Q_{n'}$? We give a negative answer to this problem by showing that if f is bad, then for any $n \geq B(f)$, $Q_n(f)$ cannot be isometrically embedded to any hypercube.

1. Introduction

Let $B = \{0, 1\}$, and $n \geq 1$ be a fixed integer. An element of B^n is called *a binary string* (or simply a string) of length n. A string $\mu \in B^n$ will be written in the coordinate form as $\mu = u_1u_2 \ldots u_n$. A string f is called a *factor* of a string μ if f appears as a sequence of $|f|$ consecutive bits of μ.

The *n-cube* Q_n (as the *hypercube* is also called) is the graph defined on the vertex set B^n, two vertices being adjacent if they differ in exactly one coordinate. Hypercube plays an important role in many areas of discrete mathematics and computer science. An excellent survey on hypercubes can be found in [11]. Clearly, $|V(Q_n)| = 2^n$ and Q_n has a regular degree n. However, these features limit the applications of hypercube in some fields.

To obtain additional graphs with similar properties as hypercubes but the number of vertices increasing more slowly than in hypercubes, Hsu [5] introduced *Fibonacci cubes* Γ_n as follows. The vertex set of Γ_n is the set of all strings of length n that contain no two consecutive 1s, two vertices being adjacent if they differ in exactly one coordinate. In other words, Fibonacci cubes Γ_n is the graph obtained from the hypercube Q_n by removing all the vertices that contain the factor 11. Fibonacci cubes have been extensively investigated; see the recent survey by Klavžar [9].

More generally, for an arbitrary fixed string f and $n \geq 1$, the *generalized Fibonacci cube* $Q_n(f)$ is defined as the subgraph of Q_n induced by the set of all strings of length n that do not contain f as a factor. In this notation, the Fibonacci cube Γ_n is $Q_n(11)$. Note that if $|f| > n$ then $Q_n(f) \cong Q_n$. This idea of generalized Fibonacci cubes was introduced by Ilić, Klavžar and Rho [6]. It should be pointed out that under the same name, the graphs $Q_n(1^s)$ were studied by Liu, Hsu and Chung [12], and Zagaglia Salvi [13], where 1^s means the string of length s with all bits equal to 1. Generalized Fibonacci cubes have been

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studied from different aspects, for example, in the recent papers of Azarija et al. [1–3]. In paper [1], the Wiener index of $Q_n(1^s)$ was studied; in [3], isomorphism classes of generalized Fibonacci cubes were studied and in [2], it was proved that $Q_n(f)$ is 2-connected for any string f with $|f| \geq 3$.

A natural problem on generalized Fibonacci cubes is when they embed isometrically into hypercubes. It was shown that Fibonacci cube $Q_n(11)$ is an isometric subgraph of Q_n [8]. This problem was further studied in [6], and it led to the concepts of the so called good and bad strings [10]. A string f is called *good* if $Q_n(f)$ is an isometric subgraph of Q_n for all $n \geq 1$, and otherwise *bad*. For examples, the string $f = 1^s0$ is good for all $s \geq 1$ and $f = 1^r0^2$ is bad for all $r \geq 2$. More infinite families of bad or good strings were found [6,7,14]. Klavžar and Shpectorov [10] proved that for large n the number of good strings is approximately about eight percent of all strings of the length n. They also showed that if $Q_n(f)$ is not an isometric subgraph of Q_n for some dimension n, then $Q_{n'}(f)$ is not an isometric subgraph of $Q_{n'}$ for all $n' > n$. Thus, for a bad string f there exists the smallest integer n such that $Q_n(f)$ is not an isometric subgraph of Q_n. This integer n is called *the index of f* and denoted by $B(f)$. For a good string f it can be set $B(f) = \infty$. For a bad string f, Ilić et al. [7] showed that $B(f) < |f|^2$, and further they conjectured that $B(f) < 2|f|$. This conjecture was proved by Wei and Zhang [15].

Suppose that $Q_n(f)$ is not an isometric subgraph of Q_n. It would still be possible that $Q_n(f)$ embeds isometrically into some $Q_{n'}$, where $n' \geq n$. Hence, Ilić, Klavžar and Rho posed the following problem:

Problem 1.1 ([6]). Suppose that $Q_n(f)$ is not an isometric subgraph of Q_n. Is there a dimension n' such that $Q_n(f)$ can be isometrically embedded into $Q_{n'}$?

With respect to Problem 1.1, Ilić, Klavžar and Rho were inclined to believe that the answer is negative in most (if not all) cases, and they also showed an example of the string $f = 101$ to confirm their guessing. It is proved that the answer to Problem 1.1 is indeed negative by the following theorem:

Theorem 1.2. *Let f be a bad string. Then for all $n \geq B(f)$, $Q_n(f)$ cannot be isometrically embedded into any hypercube.*

Note that if $Q_n(f)$ ($n \geq B(f)$) cannot be isometrically embedded into any hypercube, then f must be a bad string. Thus we have the following result:

Corollary 1.3. *The string f is bad if and only if there exists an integer k and for any $n \geq k$, $Q_n(f)$ cannot be isometrically embedded into any hypercube.*

The rest of this paper is organized as follows. In Section 2 several concepts and results needed in this paper are introduced, and in Section 3 we prove Theorem 1.2 and thus the answer to Problem 1.1 is given.

2. Preliminaries

For a string $f = f_1f_2 \cdots f_d$ let $f^R = f_d \cdots f_2f_1$ be the *reverse* of f and $\overline{f} = \overline{f}_1\overline{f}_2 \cdots \overline{f}_d$ be the binary *complement* of f, where $\overline{f}_i = 1 - f_i$, $i = 1, 2, \ldots, d$. Let $\cong$ stand for graph isomorphism. Ilić et al. got the following result:

Lemma 2.1 ([6]). *Let f be a string and $n \geq 1$. Then $Q_n(f) \cong Q_n(\overline{f}) \cong Q_n(f^R)$.*

Given two strings ζ and ξ with the same length, their *Hamming distance* $H(\zeta, \xi)$ is the number of coordinates in which they differ. For a connected graph G, the *distance* $d_G(\mu, \nu)$ between vertices μ and ν is the length of a shortest μ, ν-path. It is well-known that for any vertices $\zeta, \xi \in V(Q_n)$, $d_{Q_n}(\zeta, \xi) = H(\zeta, \xi)$.

Obviously, for any subgraph G' of G, $d_{G'}(\mu, \nu) \geq d_G(\mu, \nu)$. If $d_{G'}(\mu, \nu) = d_G(\mu, \nu)$ for all $\mu, \nu \in V(G')$, then G' is called an *isometric subgraph* of G, and simply write $G' \hookrightarrow G$, and $G' \not\hookrightarrow G$ otherwise. More generally, let G' and G be any arbitrary graphs. A mapping $h : V(G') \to V(G)$ is an *isometric embedding* of G' into G if $d_{G'}(\mu, \nu) = d_G(h(\mu), h(\nu))$ for any $\mu, \nu \in V(G')$.

For two vertices μ and ν of a graph G, the set of vertices lying on shortest μ, ν-paths is called *the interval between μ and ν*, denoted by $I_G(\mu, \nu)$. Let f be a given string, $\alpha, \beta \in V(Q_n(f))$ and $p \geq 2$. Then α and β are called *p-critical strings* [6] for $Q_n(f)$ if $d_{Q_n}(\alpha, \beta) = p$, but none of the neighbors of α (or β) in $I_{Q_n}(\alpha, \beta)$ belongs to $Q_n(f)$.

The following Lemma 2.2 holds for p-critical strings, and it gives a tool to be used to prove whether $Q_n(f)$ is an isometric subgraph of Q_n. Note that the sufficiency of this lemma was proved by Ilić et al. [6].

Lemma 2.2 ([15]). *Suppose that f is a binary string and $n \geq 1$. Then $Q_n(f) \not\hookrightarrow Q_n$ if and only if there exist p-critical strings for $Q_n(f)$ for some $p \geq 2$.*

Wei and Zhang studied p-critical strings for $Q_{B(f)}(f)$, and shown the following result (see Theorem 3.2, p. 421 in [15]):

Theorem 2.3 ([15]). *If there exist p-critical strings for $Q_{B(f)}(f)$, then $p = 2$ or $p = 3$.*

Let $\alpha = a_1a_2 \cdots a_{B(f)}$ and $\beta = b_1b_2 \cdots b_{B(f)}$ be p-critical strings for $Q_{B(f)}(f)$. Then α and β also can be constructed uniquely based on p copies of f, as shown in Corollary 2.5. To introduce this result, we need Note 2.4, which follows from the proof of Theorem 2.3. This note will be referenced after several notations [7,10] being explained.