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Degree sum conditions for path-factors with specified end vertices in bipartite graphs

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ABSTRACT

Let G be a graph, and let S be a subset of the vertex set of G. We denote the set of the end vertices of a path P by $end(P)$. A path P is an S-path if $|V(P)| \geq 2$ and $V(P) \cap S = end(P)$. An S-path-system is a graph H such that H contains all vertices of S and every component of H is an S-path. In this paper, we give a sharp degree sum condition for a bipartite graph to have a spanning S-path-system.

1. Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph, and let S be a subset of the vertex set of G. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. A spanning subgraph H of G is called a *path-factor* if every component of H is a path of order at least two. Akiyama, Avis and Era [1] gave a necessary and sufficient condition for the existence of path-factors. They proved that a graph G has a path-factor if and only if $i(G \setminus X) \leq 2|X|$ for all $X \subseteq V(G)$, where $i(G \setminus X)$ denotes the number of isolated vertices in $G \setminus X$. We denote the set of the end vertices of a path P by $end(P)$. A path P is an *S-path* if $|P| \geq 2$ and $V(P) \cap S = end(P)$. An S-path-system is a graph H such that H contains all vertices of S and every component of H is an S-path. Gallai [2] gave a necessary and sufficient condition for the existence of S-path-systems. He proved that for a graph G and a subset S of $V(G)$ with even order, G has an S-path-system if and only if $|S|/2 \leq |X| + \sum_{D \in cmp(G \setminus X)} \lfloor |D \cap S|/2 \rfloor$ for all $X \subseteq V(G)$, where $cmp(G \setminus X)$ denotes the set of components in $G \setminus X$. (In fact, he gave a min–max formula.) On the other hand, a spanning S-path-system can be regarded as a path-factor with specified end vertices. It is known that the problem of determining whether a given graph has a spanning S-path-system is NP-complete. Therefore, we investigate degree sum conditions for the existence of spanning S-path-systems.

A spanning S-path-system is closely related to a Hamiltonian cycle passing through specified edges. In 1969, Kronk gave a degree sum condition for the existence of a Hamiltonian cycle passing through specified paths. Let $\sigma_2(G)$ be the minimum degree sum of two non-adjacent vertices of G if G is not complete; otherwise $\sigma_2(G) := \infty$. A *linear forest* is a graph in which every component is a path.

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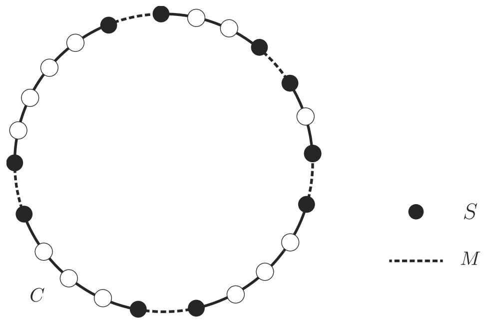

Fig. 1. A spanning S-path-system.

Theorem 1 (*Kronk [3]*). *Let G be a graph, and let F be a linear forest in G with $|E(F)| = m$. If $\sigma_2(G) \geq |G| + m$, then G has a Hamiltonian cycle passing through F.*

Recently, the fourth author [4] pointed out that it is easy to obtain the following result by using Theorem 1.

Theorem 2. *Let G be a graph and let $S \subseteq V(G)$ such that $|S| = 2k$. If $\sigma_2(G) \geq |G| + k$, then G has a spanning S-path-system.*

For the convenience of the readers, we give the proof. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S in G.

Proof. We construct a graph H from G by adding edges so that $H[S]$ has a perfect matching M. Note that $\sigma_2(H) \geq \sigma_2(G) \geq |G| + k \geq |H| + |M|$. By Theorem 1, H has a Hamiltonian cycle C passing through M. Then $C - M$ is a spanning S-path-system of G (see Fig. 1). □

In this paper, we focus on a degree sum condition for the existence of spanning S-path-systems in bipartite graphs. We denote by $G[A, B]$ a bipartite graph G with partite sets A and B. We call $G[A, B]$ with $|A| = |B|$ a *balanced* bipartite graph. For a bipartite graph $G[A, B]$, we define

$$\sigma_{1,1}(G) = \min\{d_G(a) + d_G(b) : a \in A, b \in B, ab \notin E(G)\}$$

if G is not complete; otherwise $\sigma_{1,1}(G) = \infty$.

In 2012, Zamani and West proved a bipartite version of Theorem 1.

Theorem 3 (*Zamani and West [5]*). *Let $G[A, B]$ be a balanced bipartite graph, and let F be a linear forest in G with m edges forming t_1 paths of odd length and t_2 paths of positive even length. If $\sigma_{1,1}(G) \geq (|G| + m)/2 + \epsilon$, then G has a Hamiltonian cycle passing through F, where $\epsilon = 1$ if $t_1 = 0$ or $(t_1, t_2) \in \{(1, 0), (2, 0)\}$; otherwise $\epsilon = 0$.*

By similar argument as in the proof of Theorem 2, we can prove the following theorem by using Theorem 3.

Theorem 4. *Let $G[A, B]$ be a bipartite graph such that $|A| \leq |B|$, and let S be a set of vertices such that $|S| = 2k$ and $|A \setminus S| - |B \setminus S| = |B| - |A|$.*

(1) *If $|S \cap A| = |S \cap B| = 1$, then we let $2\sigma_{1,1}(G) \geq |G| + 4 = |G| + k + 3$.*
(2) *If $|S \cap A| = 0$ or $|S \cap A| = |S \cap B| = 2$, then we let $2\sigma_{1,1}(G) \geq |G| + k + 2$.*
(3) *Otherwise, we let $2\sigma_{1,1}(G) \geq |G| + k$.*

Then G has a spanning S-path-system.

But, this degree sum condition is not best possible in the case where $|S \cap A| = |S \cap B| = 2$ and $|G| \neq 10$. It seems to be difficult to give a sharp degree sum condition by using Theorem 3. Therefore we give it without using Theorem 3.

Theorem 5. *Let k be a positive integer. Let $G[A, B]$ be a bipartite graph such that $|A| \leq |B|$, and let S be a set of vertices such that $|S| = 2k$ and $|A \setminus S| - |B \setminus S| = |B| - |A|$.*

(1) *If $|S \cap A| = |S \cap B| = 1$, then we let $2\sigma_{1,1}(G) \geq |G| + 4 = |G| + k + 3$.*
(2) *If $|S \cap A| = 0$, or $|S \cap A| = |S \cap B| = 2$ and $|G| = 10$, then we let $2\sigma_{1,1}(G) \geq |G| + k + 2$.*
(3) *Otherwise, we let $2\sigma_{1,1}(G) \geq |G| + k$.*

Then G has a spanning S-path-system.

The condition $|A \setminus S| - |B \setminus S| = |B| - |A|$ is a necessity condition for the existence of a spanning S-path-system. In Section 2, we will discuss it and the sharpness of the degree sum condition. In Section 3 and Section 4, we will give the proofs of Theorem 4 and Theorem 5, respectively.