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Refined Turán numbers and Ramsey numbers for the loose 3-uniform path of length three

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ABSTRACT

Let P denote a 3-uniform hypergraph consisting of 7 vertices a, b, c, d, e, f, g and 3 edges $\{a, b, c\}, \{c, d, e\}$, and $\{e, f, g\}$. It is known that the r-color Ramsey number for P is $R(P; r) = r + 6$ for $r \leqslant 7$. The proof of this result relies on a careful analysis of the Turán numbers for P. In this paper, we refine this analysis further and compute, for all n, the third and fourth order Turán numbers for P. With the help of the former, we confirm the formula $R(P; r) = r + 6$ for $r \in \{8, 9\}$.

1. Introduction

For brevity, 3-uniform hypergraphs will be called here *3-graphs*. Given a family of 3-graphs $\mathcal{F}$, we say that a 3-graph H is *$\mathcal{F}$-free* if for all $F \in \mathcal{F}$ we have $H \not\supseteq F$.

For a family of 3-graphs $\mathcal{F}$ and an integer $n \geqslant 1$, the *Turán number of the 1st order*, that is, the ordinary Turán number, is defined as

$$\mathrm{ex}^{(1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

Every n-vertex $\mathcal{F}$-free 3-graph with $\mathrm{ex}^{(1)}(n; \mathcal{F})$ edges is called 1-*extremal for* $\mathcal{F}$. We denote by $\mathrm{Ex}^{(1)}(n; \mathcal{F})$ the family of all, pairwise non-isomorphic, n-vertex 3-graphs which are 1-extremal for $\mathcal{F}$. Further, for an integer $s \geqslant 1$, *the Turán number of the* $(s+1)$-*st order* is defined as

$$\mathrm{ex}^{(s+1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n,\ H \text{ is } \mathcal{F}\text{-free, and } \forall H' \in \mathrm{Ex}^{(1)}(n; \mathcal{F}) \cup \cdots \cup \mathrm{Ex}^{(s)}(n; \mathcal{F}), H \not\subseteq H'\},$$

if such a 3-graph H exists. Note that if $\mathrm{ex}^{(s+1)}(n; \mathcal{F})$ exists then, by definition,

$$\mathrm{ex}^{(s+1)}(n; \mathcal{F}) < \mathrm{ex}^{(s)}(n; \mathcal{F}). \tag{1}$$

An n-vertex $\mathcal{F}$-free 3-graph H is called $(s+1)$-*extremal for* $\mathcal{F}$ if $|E(H)| = \mathrm{ex}^{(s+1)}(n; \mathcal{F})$ and $\forall H' \in \mathrm{Ex}^{(1)}(n; \mathcal{F}) \cup \cdots \cup \mathrm{Ex}^{(s)}(n; \mathcal{F}), H \not\subseteq H'$; we denote by $\mathrm{Ex}^{(s+1)}(n; \mathcal{F})$ the family of n-vertex 3-graphs which are $(s+1)$-extremal for $\mathcal{F}$. In the case when $\mathcal{F} = \{F\}$, we will write F instead of $\{F\}$.

A loose 3-*uniform path of length* 3 is a 3-graph P consisting of 7 vertices, say, a, b, c, d, e, f, g, and 3 edges $\{a, b, c\}, \{c, d, e\}$, and $\{e, f, g\}$. The *Ramsey number* $R(P; r)$ is the least integer n such that every r-coloring of the edges of the complete 3-graph K_n results in a monochromatic copy of P. Gyárfás and Raeisi [6] proved, among many other results, that $R(P; 2) = 8$. (This result was later extended to loose paths of arbitrary lengths, but still $r = 2$, in [13].) Then Jackowska [9] showed that $R(P; 3) = 9$ and $r + 6 \leqslant R(P; r)$ for all $r \geqslant 3$. In turn, in [10] and [11], Turán numbers of the first and second order, $\mathrm{ex}^{(1)}(n; P)$

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and $\mathrm{ex}^{(2)}(n; P)$, have been determined for all feasible values of n, as well as the single third order Turán number $\mathrm{ex}^{(3)}(12; P)$. Using these numbers, in [11], we were able to compute the Ramsey numbers $R(P; r)$ for $r = 4, 5, 6, 7$.

Theorem 1 (*[6,9,11]*). *For all $r \leqslant 7$, $R(P; r) = r + 6$.*

In this paper we determine, for all $n \geqslant 7$, the Turán numbers for P of the third and the fourth order, $\mathrm{ex}^{(3)}(n; P)$ and $\mathrm{ex}^{(4)}(n; P)$. The former allows us to compute two more Ramsey numbers.

Theorem 2. *For all $r \leqslant 9$, $R(P; r) = r + 6$.*

It seems that in order to make a further progress in computing the Ramsey numbers $R(P; r)$, $r \geqslant 10$, one would need to determine higher order Turán numbers $\mathrm{ex}^{(s)}(n; P)$, at least for some small values of n. Unfortunately, the fourth order numbers are not good enough.

Throughout, we denote by S_n the 3-graph on n vertices and with $\binom{n-1}{2}$ edges, in which one vertex, referred to as *the center*, forms edges with all pairs of the remaining vertices. Every sub-3-graph of S_n without isolated vertices is called *a star*, while S_n itself is called *the full star*. We denote by C *the triangle*, that is, a 3-graph with six vertices a, b, c, d, e, f and three edges $\{a, b, c\}$, $\{c, d, e\}$, and $\{e, f, a\}$. Finally, M stands for a pair of disjoint edges.

In the next section we state all, known and new, results on ordinary and higher order Turán numbers for P, including Theorem 9 which provides a complete formula for $\mathrm{ex}^{(3)}(n; P)$. We also define conditional Turán numbers and quote from [11] three useful lemmas about the conditional Turán numbers with respect to P, C, M. Then, in Section 3, we prove Theorem 2, while the remaining sections are devoted to proving Theorem 9.

2. Turán numbers

A celebrated result of Erdős, Ko, and Rado [2] asserts that for $n \geqslant 6$, $\mathrm{ex}^{(1)}(n; M) = \binom{n-1}{2}$. Moreover, for $n \geqslant 7$, $\mathrm{Ex}^{(1)}(n; M) = \{S_n\}$. We will need the second order version of this Turán number, together with the 2-extremal family. Such a result has been proved already by Hilton and Milner [8, Theorem 3, $s = 1$] (see [4] for a simple proof). For a given set of vertices V, with $|V| = n \geqslant 7$, let us define two special 3-graphs. Let $x, y, z, v \in V$ be four different vertices of V. We set

$$G_1(n) = \{\{x, y, z\}\} \cup \left\{h \in \binom{V}{3} : v \in h, h \cap \{x, y, z\} \neq \emptyset\right\},$$

$$G_2(n) = \{\{x, y, z\}\} \cup \left\{h \in \binom{V}{3} : |h \cap \{x, y, z\}| = 2\right\}.$$

Note that for $i \in \{1, 2\}$, $G_i(n) \not\supset M$ and $|G_i(n)| = 3n - 8$.

Theorem 3 (*[8]*). *For $n \geqslant 7$, $\mathrm{ex}^{(2)}(n; M) = 3n - 8$ and $\mathrm{Ex}^{(2)}(n; M) = \{G_1(n), G_2(n)\}$.*

Later, we will also use the fact that $C \subset G_i(n) \not\supset P$, $i = 1, 2$.

Recently, the third order Turán number for M has been established by Han and Kohayakawa. Let $G_3(n)$ be the 3-graph on n vertices, with distinguished vertices x, y_1, y_2, z_1, z_2 whose edge set consists of all edges spanned by x, y_1, y_2, z_1, z_2 except for $\{y_1, y_2, z_i\}$, $i = 1, 2$, and all edges of the form $\{x, z_i, v\}$, $i = 1, 2$, where $v \notin \{x, y_1, y_2, z_1, z_2\}$. Note that $|G_3(n)| = 8 + 2(n - 5) = 2n - 2$.

Theorem 4 (*[7, Theorem 1.6]*). *For $n \geqslant 7$, $\mathrm{ex}^{(3)}(n; M) = 2n - 2$ and $\mathrm{Ex}^{(3)}(n; M) = \{G_3(n)\}$.*

Interestingly, the number $\binom{n-1}{2}$ serves as the Turán number for two other 3-graphs, C and P. The Turán number $\mathrm{ex}^{(1)}(n; C)$ has been determined in [3] for $n \geqslant 75$ and later for all n in [1].

Theorem 5 (*[1]*). *For $n \geqslant 6$, $\mathrm{ex}^{(1)}(n; C) = \binom{n-1}{2}$. Moreover, for $n \geqslant 8$, $\mathrm{Ex}^{(1)}(n; C) = \{S_n\}$.*

For large n, the Turán numbers for longer (than three) loose 3-uniform paths were found in [12]. The case of length three has been omitted in [12], probably because the authors thought it had been taken care of in [5], where k-uniform loose paths were considered, $k \geqslant 4$. However, the method used in [5] did not quite work for 3-graphs. In [10] we fixed this omission. Given two 3-graphs F_1 and F_2, by $F_1 \cup F_2$ denote a vertex-disjoint union of F_1 and F_2. If $F_1 = F_2 = F$ we will sometimes write $2F$ instead of $F \cup F$.

Theorem 6 (*[10]*).

$$\mathrm{ex}^{(1)}(n; P) = \begin{cases} \binom{n}{3} & \text{and} \quad \mathrm{Ex}^{(1)}(n; P) = \{K_n\} \quad \text{for } n \leqslant 6\,, \\ 20 & \text{and} \quad \mathrm{Ex}^{(1)}(n; P) = \{K_6 \cup K_1\} \quad \text{for } n = 7\,, \\ \binom{n-1}{2} & \text{and} \quad \mathrm{Ex}^{(1)}(n; P) = \{S_n\} \quad \text{for } n \geqslant 8\,. \end{cases}$$