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Adjacent vertex distinguishing total coloring of graphs with maximum degree 4

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ABSTRACT

Let k be a positive integer. An adjacent vertex distinguishing (for short, AVD) total k-coloring ϕ of a graph G is a proper total k-coloring of G such that no pair of adjacent vertices have the same set of colors, where the set of colors at a vertex v is $\{\phi(v)\} \cup \{\phi(e) : e \text{ is incident to } v\}$. Zhang et al. conjectured in 2005 that every graph with maximum degree Δ has an AVD total $(\Delta + 3)$-coloring. Recently, Papaioannou and Raftopoulou confirmed the conjecture for 4-regular graphs. In this paper, by applying the Combinatorial Nullstellensatz, we verify the conjecture for all graphs with maximum degree 4.

1. Introduction

All graphs considered in this paper are simple and undirected. We follow the standard notation and terminology as can be found in [6]. Let $G = (V(G), E(G))$ be a graph and $T(G) = V(G) \cup E(G)$. For a vertex $x \in V(G)$, we use $N_G(v)$ and $E_G(v)$ to denote the set of vertices adjacent to v and the set of edges incident to v, respectively. An ℓ-vertex or ℓ^--vertex of G is a vertex of degree ℓ or at most ℓ, respectively. Let $V_\ell(G)$ and $V_{\ell^-}(G)$ be the sets of ℓ-vertices and ℓ^--vertices, respectively, in G. We also use V_ℓ and V_{ℓ^-} for short if the graph G is understood in context. The maximum degree of G is denoted by $\Delta(G)$.

Let k be a positive integer and $[k] = \{1, 2, \ldots, k\}$. A mapping $\phi : T(G) \to [k]$ is a proper total k-coloring if, for any two adjacent or incident elements $z_1, z_2 \in T(G)$, it is $\phi(z_1) \neq \phi(z_2)$. Let $C_\phi(v) = \{\phi(v)\} \cup \{\phi(e) : e \in E_G(v)\}$ and $m_\phi(v) = \phi(v) + \sum_{e\in E_G(v)}\phi(e)$ for any vertex $v \in V(G)$. A proper total k-coloring ϕ of G is adjacent vertex distinguishing (for short, AVD) if $C_\phi(u) \neq C_\phi(v)$ whenever $uv \in E(G)$. The AVD total chromatic number $\chi_a^t(G)$ is the smallest integer k such that G has an AVD total k-coloring.

The AVD total coloring is related to vertex-distinguishing edge coloring which requires that every pair of vertices receives different the sets of colors. The vertex-distinguishing edge coloring was introduced by Burris and Schelp [7], and independently by Černý et al. [9] (under the notion of observability). This type of coloring has been well studied over the last decade (see, for example, [2–5]). It was later extended to require only adjacent vertices to be distinguished by Zhang et al. [14], which was in turn extended to total coloring [13].

Zhang et al. [13] determined $\chi_a^t(G)$ for some basic graphs such as complete graphs and complete bipartite graphs and made the following conjecture.

Conjecture 1.1 ([13]). *For any graph G, $\chi_a^t(G) \leq \Delta(G) + 3$.*

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Chen [8] and Wang [12], independently, confirmed Conjecture 1.1 for graphs with maximum degree 3. Later, Hulgan [10] presented a concise proof on this result. Recently Papaioannou and Raftopoulou [11] verified Conjecture 1.1 for 4-regular graphs.

Theorem 1.2 (*[11]*). *For any 4-regular graph G, $\chi_a^t(G) \le 7$.*

The aim of this paper is to extend Theorem 1.2 from 4-regular graphs to graphs with maximum degree 4. We prove the following result.

Theorem 1.3. *For any graph G with maximum degree 4, $\chi_a^t(G) \le 7$.*

We use a polynomial method based on the Combinatorial Nullstellensatz due to Alon [1]. In fact, in Section 3, we prove a stronger result as follows.

Theorem 1.4. *Every graph with maximum degree 4 has a proper total 7-coloring satisfying:*

(i) *For any two adjacent 4-vertices u and v, $C_\phi(u) \ne C_\phi(v)$;*
(ii) *For any two adjacent 3^--vertices u and v, $m_\phi(u) \ne m_\phi(v)$.*

Remark. If $m_\phi(u) \ne m_\phi(v)$, then the sets of colors must be different. Also, in the definition of AVD total coloring it requires that any two adjacent vertices have different color sets. Theorem 1.4 does not cover a 4-vertex that is adjacent to a 3^--vertex. Of course, in a proper total coloring, two adjacent vertices of different degrees also have different color sets.

2. A polynomial associated with AVD total coloring

Let G be a graph with maximum degree 4 and H be an induced subgraph in $G[V_{3^-}]$. An *H-partial AVD total 7-coloring* of G is a mapping $\phi : T(G) - T(H) \to [7]$, satisfying the following two conditions:

(a) For any two adjacent or incident elements $z_1, z_2 \in T(G) - T(H)$, $\phi(z_1) \ne \phi(z_2)$;
(b) For $uv \in E(G - H)$, $C_\phi(u) \ne C_\phi(v)$ if $d_G(u) = d_G(v) = 4$, and $m_\phi(u) \ne m_\phi(v)$ if $d_G(u) \le 3$ and $d_G(v) \le 3$.

H is called *reducible* if every H-partial AVD total 7-coloring can be extended to a proper total coloring of G satisfying the conditions of Theorem 1.4. We will use a polynomial method to prove Theorem 1.4. For this, we need the following theorem, known as the Combinatorial Nullstellensatz due to Alon [1].

Theorem 2.1 (*[1]*). *Let $\mathbb{F}$ be an arbitrary field and $P \in \mathbb{F}[x_1, \ldots, x_n]$ with degree $deg(P) = \sum_{j=1}^{n} i_j$, where each i_j is a nonnegative integer. If the coefficient of the monomial $x_1^{i_1} \ldots x_n^{i_n}$ in P is nonzero, and if $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_j| > i_j$, then there are $s_1 \in S_1, \ldots, s_n \in S_n$ such that $P(s_1, \ldots, s_n) \ne 0$.*

Let H be an induced subgraph of $G[V_{3^-}]$. Denote $V(H) = \{v_1, \ldots, v_h\}$ and $E(H) = \{e_1, \ldots, e_k\}$. Each element $z \in T(H) = V(H) \cup E(H)$ is associated with a variable x_z. Let D be an arbitrary orientation of H and ϕ be an H-partial AVD total 7-coloring of G. For each vertex $v \in V(H)$, $N_D^+(v)$ is the set of arcs with v as the initial vertex. For each vertex $v \in V(H)$, let $\mu_H(v) = x_v + \sum_{e \in E_H(v)} x_e$ and

$$
\begin{aligned}
\mathcal{P}_{D,\phi}(H; v) = & \prod_{u \in N_G(v) \setminus V(H)} \left((x_v - \phi(u))(x_v - \phi(uv)) \prod_{e \in E_H(v)} (x_e - \phi(uv)) \right) \\
& \cdot \prod_{u \in (V_{3^-} \cap N_G(v)) \setminus V(H)} \left(\mu_H(v) + \sum_{e \in E_G(v) \setminus E_H(v)} \phi(e) - \phi(u) - \sum_{e \in E_G(u)} \phi(e) \right) \\
& \cdot \prod_{u \in N_D^+(v)} (x_v - x_u) \left(\mu_H(v) + \sum_{e \in E_G(v) \setminus E_H(v)} \phi(e) - \mu_H(u) - \sum_{e \in E_G(u) \setminus E_H(u)} \phi(e) \right) \\
& \cdot \prod_{e \in E_H(v)} (x_v - x_e) \prod_{\substack{e_i, e_j \in E_H(v) \\ i<j}} (x_{e_i} - x_{e_j}).
\end{aligned}
$$

Remark. In $\mathcal{P}_{D,\phi}(H; v)$, the first product assures that v and every edge $e \in E_H(v)$ would have different colors than its incident elements in $T(G) - T(H)$; while the last two products (together with some parts of the third product) assure that v and every edge $e \in E_H(v)$ would have different color than its incident elements in $T(H)$. Moreover, the second and third products guarantee $m_\phi(u) \ne m_\phi(v)$ for any vertex $u \in V_{3^-} \cap N_G(v)$.