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Ramsey numbers for degree monotone paths

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ABSTRACT

A path $v_1, v_2, \ldots, v_m$ in a graph G is *degree-monotone* if $deg(v_1) \leq deg(v_2) \leq \cdots \leq deg(v_m)$ where $deg(v_i)$ is the degree of v_i in G. Longest degree-monotone paths have been studied in several recent papers. Here we consider the Ramsey type problem for degree monotone paths. Denote by $M_k(m)$ the minimum number M such that for all $n \geq M$, in any k-edge coloring of K_n there is some $1 \leq j \leq k$ such that the graph formed by the edges colored j has a degree-monotone path of order m. We prove several nontrivial upper and lower bounds for $M_k(m)$.

1. Introduction

A path $v_1, v_2, \ldots, v_m$ in a graph G is *degree-monotone* if $deg(v_1) \leq deg(v_2) \leq \cdots \leq deg(v_m)$ where $deg(v_i)$ is the degree of v_i in G. The maximum order over all degree-monotone paths in G is denoted by $mp(G)$. General monotone path problems were systematically treated long ago by Chvatal and Komlós [9], who related oriented graphs and oriented paths to various path monotonicity problems, motivated by the famous Erdős–Szekeres Theorem [11,13] on monotone sub-sequences, and by the Gallai–Hasse–Roy–Vitaver Theorem (see [20]). Another famous monotone path problem is suggested by Graham and Kleitman [15] in which the edges of K_n are bijectively labeled by $[1, \ldots, \binom{n}{2}]$ and the problem is to determine the minimum over all possible labelings of a maximum monotone path.

The study of degree monotone paths and $mp(G)$ was explicitly suggested and developed in connection with certain domination problems by Deering et al. [10] and further developed by Caro et al. [5,6] who studied $mp(G)$ and related parameters in the context of extremal Turán type results.

One important observation which is immediate from the Gallai–Hasse–Roy–Vitaver Theorem is that $mp(G) \geq \chi(G)$. Indeed, if we orient an edge from a low degree vertex to a high degree vertex (breaking ties arbitrarily), then a directed path in the resulting oriented graph corresponds to a degree-monotone path in the original undirected graph, and the Gallai–Hasse–Roy–Vitaver Theorem asserts that in any orientation, the order of a longest directed path is at least as large as the chromatic number. Hence $mp(G)$ is a nontrivial upper bound for the chromatic number, which is sometimes tight.

In Ramsey theory, some interesting and active research is about $R(P_1, \ldots, P_k)$, the Ramsey number for k-edge-colored complete graphs that forces a monochromatic path P_j in the edges colored j , for some $1 \leq j \leq k$ (see for example [17,19]). In this paper we study the corresponding Ramsey type problem for monotone paths where monotonicity is determined by the most basic parameter, the degree of a vertex. A formal definition follows.

A *k-edge coloring* is a coloring of the edges of a graph where each edge is given one of k distinct colors. Denote by $M = M(m_1, m_2, \ldots, m_k)$ the minimum number M such that for all $n \geq M$, in any k-edge coloring of K_n, for some j where $1 \leq j \leq k$, the spanning monochromatic graph G_j formed by the edges colored j satisfies $mp(G_j) \geq m_j$. In the diagonal case

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$m = m_1 = \cdots = m_k$, we write $M_k(m)$. We refer to a monochromatic degree-monotone path in this context as an *mdm-path* for short. We will always assume that $k \geq 2$ and $m \geq 3$ and (in the non-diagonal case) $m_i \geq 3$ for all $i = 1, \ldots, k$ to avoid the trivial cases.

As we shall see, an upper bound for $M(m_1, \ldots, m_k)$ can be obtained via some classical techniques (summarized in Lemma 2.1) developed around the multicolor version of the famous Nordhaus–Gaddum Theorem [14,18]. However, in several cases this upper bound is not sharp, and getting better upper bounds seems as a highly non-trivial task requiring new ideas, among them some characterization of certain bipartite graphs with a constrained degree sequence. Also, we may not assume monotonicity in the sense explained in the following paragraph, hence to get a lower bound construction we have to overcome this difficulty. The open problems mentioned in the end of the paper indicate the various interesting directions opened by the Ramsey degree-monotone path problem.

One should observe a subtlety in the definitions of $M_k(m)$ (as well as $M(m_1, \ldots, m_k)$). It is not clear that if n is the smallest integer for which K_n satisfies the stated property, then $M_k(m) = n$. This is because being true for n, does not a priori imply it for $n + 1$ as the parameter $mp(G)$ is not hereditary. For example, $mp(K_{2,3}) = 2$ whereas for its induced subgraph $K_{2,2}$ we have $mp(K_{2,2}) = 4$. This issue occurs in the setting of edge colorings of K_n as well. Consider a 2-edge coloring of K_5 with color 1 inducing a $K_{2,3}$. Then there is no monotone path of order 4 in any of the colors, while the colored K_4 subgraph obtained by removing a vertex incident with two edges of color 1 has a monotone path of order 4 in color 1. Hence the requirement in the definition that M is the smallest integer such that for *all* $n \geq M$ the stated property holds, is important. These sort of Ramsey-degree problems (with the related subtle monotonicity problem just mentioned) originated in some papers by Albertson [1,2] and Albertson and Berman [3], and were further developed shortly afterward by Chen and Schelp [7] and Erdős et al. [12]. We mention the following interesting result that appeared in [12].

Theorem 1.1. *In any 2-coloring of the edges of K_n, where $n \geq R(m, m)$, there is a monochromatic copy of K_m with vertices $v_1, \ldots, v_m$ such that in the host monochromatic graph G,*

$$\max\{deg(v_i) : i = 1, \ldots, m\} - \min\{deg(v_i) : i = 1, \ldots, m\} \leq R(m, m) - 2,$$

and this is sharp for $n \geq 4(r - 1)(r - 2)$ where $r = R(m, m)$.

Having all these facts in mind we are now ready to state our first main result, which provides general upper and lower bounds for $M_k(m)$.

Theorem 1.2. *Let $k \geq 2$ and $m \geq 3$ be integers. Then:*

$$\frac{(m-1)^k}{2} + \frac{m-1}{2} + 1 \leq M_k(m) \leq (m-1)^k + 1.$$

In fact, more generally, if $m_i \geq 3$ for all $i = 1, \ldots, k$, then $M(m_1, \ldots, m_k) \leq \prod_{i=1}^{k}(m_i - 1) + 1$.

Notice that the upper and lower bounds for $M_k(m)$ differ by a factor smaller than 2.

As usual in most Ramsey type problems, proving tighter bounds, or even computing exact small values, turns out to be a difficult task already in the first, and perhaps most interesting, case of paths of order 3, namely $M_k(3)$. This case can also be interpreted as requiring that the degree of every vertex of a graph with no isolated edges is a local extremum (either strictly smaller than the degree of all its neighbors or strictly larger than the degree of all its neighbors). Observe that Theorem 1.2 gives $2^{k-1} + 2 \leq M_k(3) \leq 2^k + 1$. Our next theorem improves both upper and lower bounds.

Theorem 1.3. $M_2(3) = 4$, $M_3(3) = 8$ *and* $\frac{3}{4}2^k + 2 \leq M_k(3) \leq 2^k - 1$ *for* $k \geq 4$.

We note that while the upper bound is only a mild improvement over the one provided by Theorem 1.2, its proof turns out to be somewhat involved.

The first off-diagonal nontrivial case is $M(3, m)$ for which we prove:

Theorem 1.4. *For all $m \geq 3$, $M(3, m) = 2(m - 1)$.*

In the rest of this paper we prove the general bounds in Section 2, the more involved tighter bounds for paths of order 3 are proved in Section 3, and the proof of Theorem 1.4 appears in Section 4. The final section contains some specific open problems. Our notation follows that of [20], and will otherwise be introduced when it first appears.

2. General upper and lower bounds

In this section we prove Theorem 1.2. The upper bound in Theorem 1.2 is a consequence of the following result proved independently by Gyárfás and Lehel [16], Bermond [4], and Chvatal [8]. They used an observation of Zykov [21] that states that in any edge coloring of a complete graph with more than $\prod_{i=1}^{k}(m_i - 1)$ vertices with k colors, there is a color i that induces a graph whose chromatic number is at least m_i, together with the Gallai–Hasse–Roy–Vitaver Theorem to deduce: